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Nonuniform circular ensembles

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We consider circular ensembles with nonuniform weight functions. We investigate the universality of short-range and long-range level fluctuations, which are important in the study of quantum chaotic systems. We analyze a set of hierarchic relations among the correlation functions to obtain the level density for a wide class of potentials and to demonstrate universality of correlation functions in the case of weak periodic potentials (where the term potential refers to the logarithm of the weight function). Analytic study of circular unitary ensemble is done with the help of orthogonal polynomials on the unit circle. For circular orthogonal and symplectic ensembles, we introduce skew-orthogonal polynomials on the unit circle. We consider the asymptotic forms of the polynomials for the three types of ensembles with weak potentials to give a proof of the universality. The analytic results are verified by Monte Carlo simulations of the ensembles with different weight functions. We also discuss the implications of these results in the context of conductance fluctuations in mesoscopic systems and show that the universality breaks down for strong potentials.

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I. INTRODUCTION

Random matrix theory (RMT) has diverse applications in the context of universal energy-level fluctuations in complex nuclei, atoms and molecules, model quantum chaotic systems such as billiards, kicked rotors and tops, and amorphous nanoclusters [1–6]. RMT has also been used extensively in the study of disordered and mesoscopic systems [7] where it explains the universality of conductance fluctuations. Furthermore, RMT has found applications in quantum field theory [5,8–12].

The applications of RMT can be divided in two broad classes, viz. autonomous systems and systems with time-periodic Hamiltonians or quantum maps. For chaotic autonomous systems, Hamiltonian matrices are modeled by ensembles of Hermitian matrices [1–5]. Circular ensembles, viz. ensembles of unitary matrices, are used as models for evolution operators of quantum chaotic maps [4]. Circular ensembles are also used as models for scattering matrices in chaotic systems [5,7]. As in the case of ensembles of Hermitian matrices, there are three universality classes of circular ensembles, viz. circular orthogonal ensemble (COE), circular unitary ensemble (CUE), and circular symplectic ensemble (CSE) according to the parameter values, $\beta=1,\ 2$, and 4, respectively. The three ensembles, COE, CUE, and CSE, are ensembles of symmetric unitary, general unitary, and self-dual unitary matrices and are invariant under orthogonal, unitary, and symplectic transformations, respectively.

The primary interest in these ensembles is in the study of energy level fluctuations. In the Hermitian cases, Gaussian as well as non-Gaussian ensembles have been studied in great detail [1,13–15]. On the other hand, for the circular ensembles most of the research is focused on ensembles with uniform weight functions as introduced by Dyson [1,7,16–18]. However, no strong justification has been given for using uniform circular ensembles. We mention that the level density has been investigated for some nonuniform CUE's in the context of lattice gauge theories [11,12] and also semiclassical theory [19]. The aim of this paper is analytic and Monte Carlo studies of all three types of ensembles of unitary matrices with nonuniform weight functions and to discuss their applications.

Circular ensembles with uniform weight function have been studied by many authors. In particular, the short-range level fluctuation properties are well understood and verified in model systems. In a recent work, we have considered the long-range two-point correlations and confirmed their validity in the system of multiply kicked rotors [20]. In this paper we investigate the short-range and long-range level fluctuations and also conductance fluctuations in nonuniform circular ensembles. We show that just like non-Gaussian ensembles of Hermitian matrices, the nonuniform circular ensembles are also exactly solvable. For $\beta=2$ this requires study of polynomials orthogonal on the unit circle [21,22]. For $\beta=1,\ 4$, orthogonal polynomials are not adequate and one needs to introduce appropriate polynomials skew orthogonal on the unit circle. Skew-orthogonal polynomials on the real line were introduced by Dyson [13] and Mehta [1,23] and recently studied for many cases [14]. We define skew-orthogonal polynomials on the unit circle in terms of which circular ensembles for $\beta=1,4$ can be solved, thereby extending the work of Szego and others [21,22]. In a few cases we have been able to work out the polynomials explicitly. For weak periodic potentials (viz., logarthim of the weight function) we confirm the universality of long-range and short-range fluctuations. A similar study of conductance fluctuations for weak periodic potentials is also consistent with the universality. However universality of conductance fluctuations breaks down for strong potentials.

In addition to polynomial methods mentioned above, we have also considered the ensembles by considering hierarchic relations among the correlation functions whereby one can derive the level density explicitly and also shed new light on universal fluctuations. Moreover, we have made an extensive Monte Carlo study of these ensembles for various

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weight functions, using methods developed for Hermitian matrices [15].

In Sec. II, we introduce circular ensembles with arbitrary weight functions. In Sec. III, we revisit the case of uniform weight function (Dyson ensemble) and give results for long-range fluctuations. In Sec. IV, we derive results for the level density in ensembles with nonuniform weight functions. In Sec. V, we analyze the hierarchic relations among the correlation functions to discuss the universal aspects of fluctuations. In Sec. VI, we give the polynomial method for $\beta=2$ for obtaining the correlation functions. In Secs. VII–IX, we introduce skew-orthogonal polynomials on the unit circle for $\beta=1,4$ ensembles. We consider in detail the polynomial results for weak periodic potentials. For $\beta=1$ and 4, we propose an ansatz for the asymptotic form of the polynomials, similar to the asymptotic results given in [21] for $\beta=2$. In Sec. X, we give Monte Carlo calculations to illustrate our results for different weight functions. Section XI concerns the conductance fluctuations which are shown to be universal in the case of weak periodic potentials, but depart from the universality for strong potentials. The results are summerized in the concluding section.

II. CIRCULAR ENSEMBLES WITH ARBITRARY WEIGHT FUNCTIONS

We consider circular ensembles with arbitrary weight function $w(\theta)$. The joint-probability density (jpd) of eigenangles $(\theta_1,\theta_2,\ldots,\theta_N)$ is given by

$$\mathcal{P}_{N,\beta}(\theta_1,\ldots,\theta_N)=c\prod_{j>k}|e^{i\theta_j}-e^{i\theta_k}|^{\beta}\prod_{l}w(\theta_l), \tag{2.1}$$

where N is the dimensionality of the matrices and c is the normalization constant. The weight function $w(\theta)$ is written sometimes in terms of a periodic function $V(\theta)$,

$$w(\theta)=e^{-\beta V(\theta)}. \tag{2.2}$$

In this form $w(\theta)$ is β dependent and $V(\theta)$ can be recognized as the potential in a Smoluchwoski process. However, in some applications below, it will be more convenient to deal with the β-independent form of $w(\theta)$. As mentioned above, the parameter β in (2.1) and (2.2) has the values 1,2,4, respectively, for matrix ensembles invariant under orthogonal, unitary, and symplectic transformations and the corresponding matrices are symmetric unitary, (general) unitary, and quaternion self-dual unitary, respectively. We recall that $\beta=1,4$ apply to the two cases of time-reversal invariant systems whereas $\beta=2$ applies to systems without time-reversal symmetry [1,16]. Note that the corresponding ensembles of unitary matrices U has a jpd proportional to $|\det[D(U)]|^2$ where $w(\theta)=|D(e^{i\theta})|^2$; see also (4.15).

The n-eigenangle or n-level density correlation function R_n (for $n=1,2,\ldots,N$) is defined by

$$R_n(\theta_1,\ldots,\theta_n)=\frac{N!}{(N-n)!}\int d\theta_{n+1}\cdots\int d\theta_N\mathcal{P}_{N,\beta}(\theta_1,\ldots,\theta_N), \tag{2.3}$$

and gives the probability density of finding n eigenangles at $\theta_1,\ldots,\theta_n$, irrespective of the positions of the remaining eigenangles. For large N, the mean eigenangle spacing is given in terms of the level density $R_1(\theta)$ as $D(\theta)=[R_1(\theta)]^{-1}$. To describe the eigenangle fluctuations we first unfold the spectrum by

$$r_j=\int_0^{\theta_j}R_1(\phi)d\phi. \tag{2.4}$$

Then

$$\mathbf{R}_n(r_1,\ldots,r_n)=\frac{R_n(\theta_1,\ldots,\theta_n)}{R_1(\theta_1)\cdots R_1(\theta_n)} \tag{2.5}$$

is the n-level correlation function for the unfolded spectra. Note that $\mathbf{R}_1(r)=1$.

We remark here that (2.5) is usually considered in the limit $N\to\infty$ with fixed values of all the difference variables (r_j-r_k). In this case the $\mathbf{R}_n$ describe the short-range fluctuations. For long-range fluctuations [20], we consider difference variables $(r_j-r_k)\simeq N/2$ for large N. We will give results for both cases.

Finally, we mention that we will use the notation $\langle A\rangle$ for $N^{-1}\operatorname{tr}A$, $\bar{F}$ for the ensemble average of F and f^* for the complex conjugate of f.

III. RESULTS FOR DYSON ENSEMBLE

We first give results for the potential-free case (i.e., $V=0$). In this case Dyson [1,17] has derived R_n for all n and N and has given the short-range version of the unfolded correlations $\mathbf{R}_n$. With minor changes, we can write $\mathbf{R}_n$ applicable for short-range as well as long-range fluctuations. Since level density is a constant,

$$R_1(\theta)=\frac{N}{2\pi},\quad -\pi<\theta<\pi, \tag{3.1}$$

the corresponding unfolding function is

$$r_j=\frac{N\theta_j}{2\pi}. \tag{3.2}$$

The unfolded correlation function is given, for all three β ensembles, as a quaternion determinant,

$$\begin{aligned}\mathbf{R}_{n,\beta}(r_1,\ldots,r_n)&=\mathrm{Qdet}[\sigma_{N,\beta}(r_j-r_k)]_{j,k=1,\ldots,n}\\&=\{\det[\sigma_{N,\beta}(r_j-r_k)]\}^{1/2},\end{aligned} \tag{3.3}$$

where the last form gives the definition of the quaternion determinant (Qdet). Using the self-duality of the matrices one can write determinantlike expansions in terms of the σ_β. The σ_β in (3.3) are given by [1,17]

$$\sigma_{N,2}(r)=\begin{pmatrix}S_N(r)&0\\0&S_N(r)\end{pmatrix}, \tag{3.4}$$

$$\sigma_{N,1}(r)=\begin{pmatrix}S_N(r)&D_N(r)\\I_N(r)-\epsilon(r)&S_N(r)\end{pmatrix}, \tag{3.5}$$

$$\sigma_{N,4}(r) = \begin{pmatrix} S_{2N}(2r) & D_{2N}(2r) \\ I_{2N}(2r) & S_{2N}(2r) \end{pmatrix}, \tag{3.6}$$

where

$$D_N(r) = dS_N(r)/dr, \tag{3.7}$$

$$I_N(r) = \int_0^r S_N(r')dr', \tag{3.8}$$

$$\epsilon(r) = r/2|r|, \tag{3.9}$$

and

$$S_N(r) = \frac{\sin(\pi r)}{N \sin(\pi r/N)}. \tag{3.10}$$

The above results are valid for both long-range and short-range fluctuations. However for short-range fluctuations (i.e., fixed r and $N \to \infty$), we can simplify (3.3)–(3.10) by replacing $\sigma_{N,\beta}(r), S_N(r), D_N(r), I_N(r)$ by their $N \to \infty$ limits $\sigma_\beta(r), S(r), D(r), I(r)$, respectively, where $S(r) = \sin(\pi r)/\pi r$; for $\beta=4$ the results are expressed in terms of $S(2r)$.

As an example of above results, we consider the two-level cluster function,

$$Y_2(r) = 1 - \mathbf{R}_2(r_1, r_2), \quad r = r_1 - r_2. \tag{3.11}$$

We have

$$\begin{aligned} Y_2(r) &= [S_N(r)]^2 - D_N(r)[I_N(r) - \epsilon(r)], \quad \beta=1 \\ &= [S_N(r)]^2, \quad \beta=2 \\ &= [S_{2N}(2r)]^2 - D_{2N}(2r)I_{2N}(2r), \quad \beta=4, \end{aligned} \tag{3.12}$$

which for large $|r|$ (typically, $N/2 \geq r \gtrsim 1$) is given by [20]

$$Y_2(r) = \frac{1}{\beta N^2 \sin^2(\pi r/N)}. \tag{3.13}$$

Note also that in all three cases, (3.12) yields

$$\int_{-N/2}^{N/2} Y_2(r)dr = 1. \tag{3.14}$$

For fixed r and $N \to \infty$, (3.13) and (3.14) give back the known results

$$Y_2(r) = \frac{1}{\beta \pi^2 r^2}, \quad |r| \gtrsim 1 \tag{3.15}$$

and

$$\int_{-\infty}^{\infty} Y_2(r)dr = 1. \tag{3.16}$$

The number variance $\Sigma^2(r)$ is the variance of the number of eigenangles in intervals of length $(2\pi r/N)$ with $r>0$ and is derived from

$$\Sigma^2(r) = r - 2\int_0^r (r-s)Y_2(s)ds. \tag{3.17}$$

For $1 \lesssim r \lesssim N-1$, we find [20]

$$\begin{aligned} \Sigma^2(r) &= \frac{2}{\pi^2}\left(\ln(\tilde{r}) + \gamma + 1 - \frac{\pi^2}{8}\right), \quad \beta=1 \\ &= \frac{1}{\pi^2}[\ln(\tilde{r}) + \gamma + 1], \quad \beta=2 \\ &= \frac{1}{2\pi^2}\left(\ln(2\tilde{r}) + \gamma + 1 + \frac{\pi^2}{8}\right), \quad \beta=4, \end{aligned} \tag{3.18}$$

where $\tilde{r} = 2N \sin(\pi r/N)$, and γ is the Euler constant. See Appendix A for proof of (3.18). Note that $\Sigma^2(r) = \Sigma^2(N-r)$, as expected. For $r \ll N, \tilde{r} = 2\pi r$ giving thereby the earlier results [1,2] for short-range fluctuations.

IV. LEVEL DENSITY

In this section, we derive the level density for large N. In particular, we show that in the case of weak periodic potentials $[V(\theta) = O(1)]$ the level density $(N^{-1}R_1)$, normalized to unity, is again a constant for large N but has additional $O(N^{-1})$ corrections. For strong periodic potentials $[V(\theta) = O(N)]$, the level density departs from uniformity. As in [15], the density develops a band structure when the potential has deep minima.

The joint probability density $\mathcal{P}$ of (2.1) satisfies

$$\frac{1}{\beta}\frac{\partial \mathcal{P}}{\partial \theta_1} = \left[\frac{1}{2}\sum_{j \neq 1} \cot\left(\frac{\theta_j - \theta_1}{2}\right) - V'(\theta_1)\right]\mathcal{P}. \tag{4.1}$$

Integrating both sides of (4.1) over all variables except $(\theta_1, \ldots, \theta_n)$ we obtain an exact hierarchic set of relations linking R_{n+1} to R_n [18]. We discuss the relations for $n>1$ in the next section. Here we consider [14] the $n=1$ case,

$$\frac{1}{\beta}\frac{\partial R_1(\theta)}{\partial \theta} = \frac{1}{2}\int_{-\pi}^{\pi} R_2(\theta, \phi)\cot\left(\frac{\theta - \phi}{2}\right)d\phi - V'(\theta)R_1(\theta). \tag{4.2}$$

For large $N, R_1 = O(N)$ and $R_2(\theta, \phi) - R_1(\theta)R_1(\phi) = O(1)$. Let

$$\bar{\rho}(\theta) = N^{-1}R_1(\theta) \tag{4.3}$$

be the density normalized to unity. We also define the principal value integral

$$\bar{P}(\theta) = \frac{1}{2}\int_{-\pi}^{\pi} d\phi \bar{\rho}(\phi)\cot\left(\frac{\theta - \phi}{2}\right), \tag{4.4}$$

where (and in similar equations below) the principal value of the integral is implied. Then, to order N^{-1}, we have from (4.2),

$$\bar{P}(\theta) = \frac{1}{N}V'(\theta) + \frac{1}{\beta N}\frac{\partial}{\partial \theta}\ln \bar{\rho}(\theta), \tag{4.5}$$

valid for regions where $\bar{\rho}(\theta) \neq 0$.

To solve (4.5), we introduce the transform [18]

$$\bar{g}(\psi) = \frac{1}{2}\int_{-\pi}^{\pi} d\theta \cot\left(\frac{\psi-\theta}{2}\right)\bar{\rho}(\theta), \tag{4.6}$$

where ψ is a complex angle. Then

$$\bar{g}(\theta - i0) = \bar{P}(\theta) + i\pi\bar{\rho}(\theta), \tag{4.7}$$

so that the imaginary part of $\pi^{-1}\bar{g}(\theta-i0)$ gives back the density. We multiply both sides of (4.5) by $\bar{\rho}(\theta)[\cot(\psi-\theta)/2]$ and integrate over θ. We note first that

$$\begin{aligned}
&\frac{1}{2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\bar{\rho}(\theta)\bar{\rho}(\phi)\cot\left(\frac{\psi-\theta}{2}\right)\cot\left(\frac{\theta-\phi}{2}\right)d\theta d\phi\\
&=\frac{1}{4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\bar{\rho}(\theta)\bar{\rho}(\phi)\cot\left(\frac{\theta-\phi}{2}\right)\left[\cot\left(\frac{\psi-\theta}{2}\right)\right.\\
&\left.-\cot\left(\frac{\psi-\phi}{2}\right)\right]d\theta d\phi=\frac{1}{4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\bar{\rho}(\theta)\bar{\rho}(\phi)\left[1\right.\\
&\left.+\cot\left(\frac{\psi-\theta}{2}\right)\cot\left(\frac{\psi-\phi}{2}\right)\right]d\theta d\phi=\frac{1}{4}+[\bar{g}(\psi)]^2,
\end{aligned} \tag{4.8}$$

and

$$\int_{-\pi}^{\pi}\cot\left(\frac{\psi-\theta}{2}\right)\frac{\partial\bar{\rho}(\theta)}{\partial\theta}d\theta = 2\frac{\partial\bar{g}(\psi)}{\partial\psi}. \tag{4.9}$$

Here, in (4.8), we have symmetrized the integrand in the first step and used $(\cot A - \cot B)\cot(B-A) = 1 + \cot A \cot B$ in the second step. Equation (4.9) follows by partial integration; we use here the condition that $\bar{\rho}(\theta)=\bar{\rho}(\theta+2\pi)$ in nonbanded cases or $\bar{\rho}(\theta)=0$ at the end points in banded cases. Thus

$$\frac{1}{4}+[\bar{g}(\psi)]^2 = \frac{1}{N}\int_{-\pi}^{\pi}V'(\phi)\cot\left(\frac{\psi-\phi}{2}\right)\bar{\rho}(\phi)d\phi + \frac{2}{\beta N}\frac{\partial\bar{g}(\psi)}{\partial\psi}. \tag{4.10}$$

Now using (4.7) in (4.10), we find that the imaginary part gives back (4.5) while the real part gives

$$\begin{aligned}
[\bar{\rho}(\theta)]^2 &= \frac{1}{4\pi^2} + \frac{1}{\pi^2 N^2}[V'(\theta)]^2 - \frac{2}{\beta\pi^2 N^2}V''(\theta)\\
&\quad - \frac{1}{N\pi^2}\int_{-\pi}^{\pi}V'(\phi)\cot\left(\frac{\theta-\phi}{2}\right)\bar{\rho}(\phi)d\phi,
\end{aligned} \tag{4.11}$$

where the terms $(\{[\ln\bar{\rho}(\theta)]'\}^2 - 2[\ln\bar{\rho}(\theta)]'' + 2\beta V'(\theta)[\ln\bar{\rho}(\theta)]')/(\beta\pi N)^2$ are of order N^{-2} and have been ignored. Also, the third term on right-hand side (rhs), viz. $[2V''(\theta)/\beta\pi^2 N^2]$, is of order N^{-2} for weak potentials and N^{-1} for strong potentials and will be ignored in subsequent discussions.

When $V(\theta)=O(1)$, the case of weak periodic potentials, $\bar{\rho}(\theta)=(2\pi)^{-1}$ to the leading order. Dropping $O(N^{-2})$ terms in (4.11), we obtain

$$\begin{aligned}
\bar{\rho}(\theta) &= \frac{1}{2\pi} - \frac{1}{2\pi^2 N}\int_{-\pi}^{\pi}d\phi V'(\phi)\cot\left(\frac{\theta-\phi}{2}\right) = \frac{1}{2\pi}\\
&\quad + \frac{1}{2\pi^2 N}\frac{d}{d\theta}\int_{-\pi}^{\pi}d\phi[V(\theta)-V(\phi)]\cot\left(\frac{\theta-\phi}{2}\right),
\end{aligned} \tag{4.12}$$

giving the N^{-1}-order corrections to the density. The density in (4.12) is independent of β because of β-dependent choice of $w(\theta)$ in (2.2). To recover results for β-independent w's, we rewrite (4.12) as

$$\bar{\rho}(\theta) = \frac{1}{2\pi} + \frac{2}{\beta\pi N}\frac{d\gamma(\theta)}{d\theta}, \tag{4.13}$$

where

$$\gamma(\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi}d\phi[\ln w(\phi) - \ln w(\theta)]\cot\left(\frac{\theta-\phi}{2}\right). \tag{4.14}$$

For weight functions for which the integrals $\int_{-\pi}^{\pi}w(\theta)d\theta$ and $\int_{-\pi}^{\pi}|\ln w(\theta)|d\theta$ exist, one can write

$$w(\theta) = |D(e^{i\theta})|^2, \tag{4.15}$$

where $D(z)$ is analytic and nonzero for $|z|<1$ with $D(0)>0$ [21]. $D(z)$ is uniquely determined from $w(\theta)$. In this case (4.14) can be simplified as

$$\gamma(\theta) = \arg[D(e^{i\theta})]. \tag{4.16}$$

In later sections we will deal with the case $D(z)=[h(z)]^{-1}$ where $h(z)$ is a polynomial of order $m(\geq 0)$ with no zeros for $|z|\leq 1$. For $h(z)=1$, we obtain the Dyson ensemble. For $h(z)=z+c$ with $|c|>1$, we obtain for large N,

$$\bar{\rho}(\theta) = \frac{1}{2\pi} - \frac{2}{\beta\pi N}\left(\frac{1+\mathrm{Re}(ce^{-i\theta})}{1+|c|^2+2\,\mathrm{Re}(ce^{-i\theta})}\right). \tag{4.17}$$

As shown in Sec. VI, the result (4.17) is exact for $\beta=2$ and valid for all N. We will also consider the Jacobi class of weight functions

$$w(\theta) = (1-\cos\theta)^a(1+\cos\theta)^b, \tag{4.18}$$

where $a,b>-1/2$. In this case

$$D(z) = 2^{-(a+b)}(1-z)^a(1+z)^b \tag{4.19}$$

and

$$\begin{aligned}
\bar{\rho}(\theta) &= \frac{1}{2\pi} + \frac{a+b}{\beta\pi N} - \frac{2}{\beta N}\sum_{n=-\infty}^{\infty}\{a\delta_N(\theta-2n\pi)\\
&\quad + b\delta_N[\theta-(2n+1)\pi]\},
\end{aligned} \tag{4.20}$$

where $\delta_N(x)$ has a peak of order N at $x=0$ such that $\bar{\rho}\geq 0$ and width of order N^{-1} to ensure that its area is unity. In this case $\gamma(\theta)$ as given by (4.16) is discontinuous at the zeros of $w(\theta)$. In Sec. X we will give Monte Carlo verification of (4.17) and (4.20).

For strong potentials, say $V(\theta)=Nu(\theta)$, the third term on the rhs of (4.11) gives the $O(N^{-1})$ correction while other terms are of $O(1)$. Thus (4.14) can be written as

$$[\bar{\rho}(\theta)]^2 = \frac{1}{4\pi^2}\left[1 + 4[u'(\theta)]^2 - 4\int_{-\pi}^{\pi} u'(\phi)\cot\left(\frac{\theta-\phi}{2}\right)\bar{\rho}(\phi)d\phi\right] = \frac{1}{4\pi^2}\left[1 - 4[u'(\theta)]^2 + 4\int_{-\pi}^{\pi}[u'(\theta) - u'(\phi)]\cot\left(\frac{\theta-\phi}{2}\right)\bar{\rho}(\phi)d\phi\right], \quad (4.21)$$

correct to $O(1)$ and similar to results given earlier for non-Gaussian ensembles of Hermitian matrices [15]. Note that the rhs is taken to be zero when $\bar{\rho}(\theta)=0$, implying banded density. In some cases (4.21) can be further simplified. For example, for $u(\theta)=\alpha\cos\theta$,

$$\bar{\rho}(\theta) = \frac{1}{2\pi}(1 - 4\alpha m_1 - 4\alpha\cos\theta - 4\alpha^2\sin^2\theta)^{1/2},$$

$$\bar{m}_1 = \int_{-\pi}^{\pi}\cos(\phi)\bar{\rho}(\phi)d\phi. \quad (4.22)$$

We have developed an iterative numerical procedure for solving (4.21), which is similar to the procedure mentioned in [15]. We start with an initial guess $\bar{\rho}_0(\theta)$ which we use on the rhs of (4.21). When the rhs is negative, we set $\rho_1=0$; otherwise the rhs gives $\bar{\rho}_1^2$. Next we normalize $\bar{\rho}_1$ to unity and use it as the next guess to obtain $\bar{\rho}_2$. Iteration of this procedure gives a sequence of $\bar{\rho}_n$ which converges rapidly to the final solution of (4.21). The same method applied to (4.22) converges more rapidly. We discuss the numerical results in Sec. X.

Equation (4.12) with $V=Nu$ gives the exact solution of (4.21) in the nonbanded cases. To prove this we start with (4.5) without the $\ln\bar{\rho}$ term, i.e.,

$$\bar{P}(\theta) = u'(\theta). \quad (4.23)$$

[The $\ln\bar{\rho}$ term is $O(N^{-2})$ for weak potentials and $O(N^{-1})$ for strong potentials.] The general solution of (4.23) in the nonbanded cases is given by

$$\bar{\rho}(\theta) = A - \frac{1}{2\pi^2}\int_{-\pi}^{\pi} d\phi u'(\phi)\cot\left(\frac{\theta-\phi}{2}\right), \quad (4.24)$$

where the constant A is $1/2\pi$ from the normalization of $\bar{\rho}$, giving thereby (4.12). Equation (4.24) comes about because

$$\int_{-\pi}^{\pi} d\xi\cot\left(\frac{\theta-\xi}{2}\right)\cot\left(\frac{\phi-\xi}{2}\right) = \delta(\theta-\phi) - \frac{1}{2\pi} \quad (4.25)$$

and

$$\int_{-\pi}^{\pi}\cot\left(\frac{\theta-\xi}{2}\right)d\xi = 0. \quad (4.26)$$

Equation (4.26) follows from (4.8). To prove (4.25), note that (i) integral of both sides over θ or ϕ is zero [consistent with (4.26)], (ii) the integral becomes ∞ for $\theta=\phi$, and (iii) for $\theta \neq \phi$, (4.25) follows from the contour integral

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{dz}{z}\left(\frac{z+e^{i\theta}}{z-e^{i\theta}}\right)\left(\frac{z+e^{i\phi}}{z-e^{i\phi}}\right) = 1, \quad (4.27)$$

where the contour Γ is $|z|=1$ (anticlockwise direction) with singularities at $e^{i\theta}$ and $e^{i\phi}$ avoided.

We consider the above example $u=\alpha\cos\theta$ again with $\alpha \geq 0$. We use (4.12), or equivalently (4.24) to obtain the density in the nonbanded case. The density is positive for all θ for $0\leq\alpha<1/2$ giving thereby the range for the nonbanded case. For the banded case ($\alpha\geq 1/2$), we return to (4.22) and ask for solutions of $\bar{\rho}(0)=0$, thereby giving $\bar{m}_1$ and hence $\bar{\rho}$. We have [11]

$$\bar{\rho}(\theta) = (2\pi)^{-1}(1 - 2\alpha\cos\theta), \quad 0 \leq \alpha \leq 1/2,$$

$$= \pi^{-1}[\alpha(1-\cos\theta)(1-\alpha-\alpha\cos\theta)]^{1/2}, \quad 1/2 \leq \alpha, \quad (4.28)$$

and

$$\bar{m}_1 = -\alpha, \quad 0 \leq \alpha \leq 1/2,$$

$$= \frac{1}{4\alpha} - 1, \quad 1/2 \leq \alpha. \quad (4.29)$$

For $\alpha \gg 1, \bar{\rho}(\theta)$ is nonzero only in a narrow range near $\theta=\pi$. We finally note that the matrix jpd in this example is proportional to $|\det\exp(-\beta\alpha NU)|^2$.

V. HIERARCHIC RELATIONS

As mentioned in Sec. IV, the hierarchic relations are obtained by integrating (4.1) over all but n variables. We have, with $R_n \equiv R_n(\theta_1,\ldots,\theta_n)$ and $R_{n+1}\equiv R_{n+1}(\theta_1,\ldots,\theta_{n+1})$,

$$\frac{1}{\beta}\frac{\partial R_n}{\partial\theta_j} = \left[\frac{1}{2}\sum_{k(\neq j)}^{n}\cot\left(\frac{\theta_j-\theta_k}{2}\right) - V'(\theta_j)\right]R_n + \frac{1}{2}\int_{-\pi}^{\pi} d\theta_{n+1}R_{n+1}\cot\left(\frac{\theta_j-\theta_{n+1}}{2}\right), \quad (5.1)$$

valid for all $j=1,\ldots,n$. Using (4.3)–(4.5), we can write $V'(\theta_j)$ to $O(1)$ in regions where $R_1(\theta_j)\neq 0$, as

$$V'(\theta_j) = \frac{1}{2}\int_{-\pi}^{\pi} d\theta_{n+1}R_1(\theta_{n+1})\cot\left(\frac{\theta_j-\theta_{n+1}}{2}\right) - \frac{1}{\beta R_1(\theta_j)}\frac{\partial R_1(\theta_j)}{\partial\theta_j}, \quad (5.2)$$

and thus (5.1) can be rewritten as

$$\frac{R_1(\theta_j)}{\beta}\frac{\partial}{\partial\theta_j}\left(\frac{R_n}{R_1(\theta_j)}\right)=\frac{1}{2}\sum_{k(\neq j)}^{n}\cot\left(\frac{\theta_j-\theta_k}{2}\right)R_n+\frac{1}{2}\int_{-\pi}^{\pi}d\theta_{n+1}[R_{n+1}-R_nR_1(\theta_{n+1})]\cot\left(\frac{\theta_j-\theta_{n+1}}{2}\right). \tag{5.3}$$

Here, in the integrals (5.2) and (5.3), and similar equations below, the principal value should be taken as the integrand diverges for $\theta_{n+1}=\theta_j$. [There is no such divergence in (5.1) as R_{n+1} vanishes for $\theta_{n+1}=\theta_j$.] Now we consider the unfolded variables r_j and the corresponding correlation functions $\mathbf{R}_n$ defined in (2.4) and (2.5). Then (5.3) can be written, for $j=1,\ldots,n$, as

$$\frac{1}{\beta}\frac{\partial\mathbf{R}_n}{\partial r_j}=\frac{1}{2R_1(\theta_j)}\sum_{k(\neq j)}^{n}\cot\left(\frac{\theta_j-\theta_k}{2}\right)\mathbf{R}_n+\frac{1}{2R_1(\theta_j)}\int_{-N/2}^{N/2}dr_{n+1}[\mathbf{R}_{n+1}-\mathbf{R}_n]\cot\left(\frac{\theta_j-\theta_{n+1}}{2}\right), \tag{5.4}$$

where the θ's are functions of the corresponding r variables, and $\mathbf{R}_n\equiv\mathbf{R}_n(r_1,\ldots,r_n)$, $\mathbf{R}_{n+1}\equiv\mathbf{R}_{n+1}(r_1,\ldots,r_{n+1})$.

We consider first short-range fluctuations, viz. those described by $\mathbf{R}_n$ where all the θ_j's are in the neighborhood of, say, θ. In this case, for large N, $(\theta_j-\theta_k)R_1(\theta_j)\to(r_j-r_k)$ and $[R_1(\theta_j)]^{-1}\cot[(\theta_j-\theta_k)/2]\to 2(r_j-r_k)^{-1}$. Moreover the integral in (5.4) can be divided into two regions for θ_{n+1}—in the neighborhood of θ and outside; the second integral goes to zero since $\mathbf{R}_{n+1}-\mathbf{R}_n\to 0$ for $r_{n+1}\to\infty$. Thus, in the $N\to\infty$ limit, we find

$$\frac{1}{\beta}\frac{\partial\mathbf{R}_n}{\partial r_j}=\sum_{k(\neq j)}^{n}\frac{\mathbf{R}_n}{r_j-r_k}+\int_{-\infty}^{\infty}dr_{n+1}\frac{(\mathbf{R}_{n+1}-\mathbf{R}_n)}{r_j-r_{n+1}}, \tag{5.5}$$

valid for $j=1,\ldots,n$. Note that (5.5) becomes independent of the potential V—weak or strong and is the same as those found for Gaussian ensembles [24] as well as circular ensembles with uniform weight function [18]. Thus short-range fluctuations are universal.

For long-range fluctuations, where some or all of the pair variables (r_j-r_k) may be $O(N)$, we consider the case of weak potentials. In this case $R_1(\theta)=N/2\pi+O(1)$ and therefore (3.2) is replaced by

$$r_j=\frac{N\theta_j}{2\pi}+\frac{2}{\beta\pi}\gamma(\theta_j), \tag{5.6}$$

which follows from (2.4), (4.3), and (4.13). Now (5.4) can be written for large N as

$$\frac{1}{\beta}\frac{\partial\mathbf{R}_n}{\partial r_j}=\sum_{k(\neq j)}^{N}\frac{\pi}{N}\mathbf{R}_n\cot\left(\frac{\pi(r_j-r_k)}{N}\right)+\frac{\pi}{N}\int_{-N/2}^{N/2}dr_{n+1}[\mathbf{R}_{n+1}-\mathbf{R}_n]\cot\left(\frac{\pi(r_j-r_{n+1})}{N}\right), \tag{5.7}$$

again independent of V for $V=O(1)$. Equation (5.7) thus constitutes a proof of the universality of long-range fluctuations for the restricted class of weak potentials. Note that the long-range universality breaks down for strong potentials. Note also that (5.7) reduces to (5.5) for short-range fluctuations.

Our polynomial results in Secs. VI–IX confirm the above results for weak potentials. We can also verify directly the long-range results for all β for weak potentials by the functional-derivative method [7]. Including the self-correlation term, we have from (4.13)

$$S_2(\theta,\phi)\equiv\delta(\theta-\phi)R_1(\theta)+R_2(\theta,\phi)-R_1(\theta)R_1(\phi)=-\frac{1}{\beta}\frac{\delta R_1(\theta)}{\delta V(\phi)}=-\frac{1}{4\pi^2\beta\sin^2\left(\frac{\theta-\phi}{2}\right)}, \tag{5.8}$$

which after unfolding gives (3.13) for large $|r|$ for weak potentials. Note that (5.8) is valid also for strong potentials with nonbanded density, similar to the results for non-Gaussian ensembles [20,25].

Finally we derive some two-point long-range results from (5.8), which were given earlier [20] for $V=0$. Let

$$C_{p,q}=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}e^{ip\theta}e^{iq\phi}S_2(\theta,\phi)d\theta d\phi. \tag{5.9}$$

From (5.8), we have, for large N,

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}e^{ip\theta}e^{iq\phi}\sin^2\left(\frac{\theta-\phi}{2}\right)S_2(\theta,\phi)d\theta d\phi=-\frac{1}{\beta}\delta_{p0}\delta_{q0}+O\left(\frac{1}{N}\right), \tag{5.10}$$

so that

$$C_{p+1,q-1}-2C_{p,q}+C_{p-1,q+1}=\frac{4}{\beta}\delta_{p0}\delta_{q0}+O\left(\frac{1}{N}\right). \tag{5.11}$$

Moreover

$$C_{p,0}=C_{0,q}=0. \tag{5.12}$$

Solving (5.11) and (5.12), we obtain

$$C_{p,q}=\frac{2|p|}{\beta}\delta_{p+q,0}+O\left(\frac{1}{N}\right). \tag{5.13}$$

Equation (5.13) implies universal results for the covariance $N^2(\overline{\langle U^p\rangle\langle U^q\rangle}-\overline{\langle U^p\rangle}\,\overline{\langle U^q\rangle})$, similar to those in the non-Gaussian ensembles [20].

VI. POLYNOMIAL METHOD FOR THE UNITARY ENSEMBLES

In this section, we consider the polynomial method for deriving the correlation functions for $\beta=2$. For this, we use polynomials orthogonal on the unit circle with weight function $w(\theta)$ as defined in [21,22].

Let $\phi_\mu(z)$ be polynomial of order μ, satisfying the orthogonality condition,

$$\int_{-\pi}^{\pi} \phi_\mu(e^{i\theta})[\phi_\nu(e^{i\theta})]^* w(\theta) d\theta = g_\mu \delta_{\mu\nu}, \tag{6.1}$$

where g_μ is the normalization constant and $\mu,\nu=0,1,2,\ldots$. Let

$$\Phi_\mu(\theta) = [g_\mu^{-1} w(\theta)]^{1/2} \phi_\mu(e^{i\theta}) \tag{6.2}$$

be the corresponding orthonormal function. Then the joint probability density of (2.1) can be written for $\beta=2$ as

$$\mathcal{P}_{N,2}(\theta_1,\theta_2,\ldots,\theta_N) = \frac{1}{N!}\det[S_N(\theta_j,\theta_k)]_{j,k=1,\ldots,N}. \tag{6.3}$$

The kernel $S_N(\theta_1,\theta_2)$ is given by

$$S_N(\theta_1,\theta_2) = e^{i(N-1)(\theta_1-\theta_2)/2}\sum_{\mu=0}^{N-1}[\Phi_\mu(\theta_1)]^*\Phi_\mu(\theta_2), \tag{6.4}$$

and satisfies the integral conditions

$$\int_{-\pi}^{\pi} S_N(\theta,\theta) d\theta = N, \tag{6.5}$$

$$\int_{-\pi}^{\pi} S_N(\theta_1,\chi) S_N(\chi,\theta_2) d\chi = S_N(\theta_1,\theta_2). \tag{6.6}$$

Proof of (6.3), (6.5), and (6.6) is outlined in Appendix B. Thus Dyson's theorem [1,17] can be used to derive the correlation functions R_n. We have

$$R_{n,2}(\theta_1,\ldots,\theta_n) = \det[S_N(\theta_j,\theta_k)]_{j,k=1,\ldots,n}. \tag{6.7}$$

(Note that the quaternion determinant in this case is the ordinary determinant, since the matrix elements are quaternion scalars.) Thus, the level density and the two-point correlation function are given, respectively, by

$$R_1(\theta) = S_N(\theta,\theta), \tag{6.8}$$

$$R_2(\theta_1,\theta_2) = S_N(\theta_1,\theta_1)S_N(\theta_2,\theta_2) - S_N(\theta_1,\theta_2)S_N(\theta_2,\theta_1). \tag{6.9}$$

The sum in (6.4) is related to $\Phi_N(\theta)$ by a Christoffel-Darboux-type summation formula [21],

$$S_N(\theta_1,\theta_2) = \frac{\mathrm{Im}\{e^{-iN(\theta_1-\theta_2)/2}\Phi_N(\theta_1)[\Phi_N(\theta_2)]^*\}}{\sin[(\theta_1-\theta_2)/2]}, \tag{6.10}$$

which in the $\theta_2\to\theta_1$ limit gives the level density R_1.

We now consider some special cases. For the Dyson ensemble $[w(\theta)=1]$, we have $\phi_\mu(z)=z^\mu$ and $g_\mu=2\pi$ so that

$$S_N(\theta_1,\theta_2) = \frac{\sin[N(\theta_1-\theta_2)/2]}{2\pi\sin[(\theta_1-\theta_2)/2]}, \tag{6.11}$$

$$R_1(\theta) = S_N(\theta,\theta) = \frac{N}{2\pi}, \tag{6.12}$$

and, in terms of the unfolded variables $r=(\theta_1-\theta_2)N/2\pi$,

$$S_N(r) = \lim_{N\to\infty}\frac{S_N(\theta_1,\theta_2)}{[S_N(\theta_1,\theta_1)S_N(\theta_2,\theta_2)]^{1/2}} \tag{6.13}$$

is given by (3.10). We obtain thus the unfolded correlation functions given by (3.3), (3.4), and (3.10).

For $w(\theta)=|h(e^{i\theta})|^{-2}$ with $h(z)$ a polynomial of order m, having no zeros for $|z|\le 1$, the polynomials for $\mu\ge m$ are given by $\phi_\mu(z)=z^\mu[h(e^{i\theta})]^*$ [21]. In this case, we have [19] for $N\ge m$,

$$S_N(\theta_1,\theta_2) = \frac{\sin\left(\frac{N(\theta_1-\theta_2)}{2}+\gamma(\theta_1)-\gamma(\theta_2)\right)}{2\pi\sin[(\theta_1-\theta_2)/2]}, \tag{6.14}$$

where

$$\gamma(\theta) = -\arg[h(e^{i\theta})], \quad -\pi\le\gamma\le\pi. \tag{6.15}$$

Thus

$$R_1(\theta) = S_N(\theta,\theta) = \frac{N}{2\pi}+\frac{1}{\pi}\frac{\partial\gamma(\theta)}{\partial\theta} = \frac{N}{2\pi}-\frac{1}{2\pi}\left[e^{i\theta}\frac{h'(e^{i\theta})}{h(e^{i\theta})} + \left(e^{i\theta}\frac{h'(e^{i\theta})}{h(e^{i\theta})}\right)^*\right], \tag{6.16}$$

as given in Sec. IV; see (4.13) and (4.16). Now using the unfolded variable $r=\{N(\theta_1-\theta_2)+2[\gamma(\theta_1)-\gamma(\theta_2)]\}/2\pi$ as given by (5.6), we obtain (3.10) from (6.14) for $N\gg m$ and hence the unfolded correlation function $\mathbf{R}_n$ for $\beta=2$ as in (3.3), (3.4), and (3.10). Note that $R_1(\theta)$ is a constant to leading order (as for the $V=0$ case), but the $O(1)$ correction is required for the long-range correlations.

For the Jacobi class of weight functions (4.18), the ϕ_μ can be written [21,22] in terms of the Jacobi polynomials $P_\nu^{(a-1/2,b-1/2)}(\cos\theta)$ and $P_{\nu-1}^{(a+1/2,b+1/2)}(\cos\theta)$, where $\nu=\mathrm{Int}[(\mu+1)/2]$. Now using the asymptotic forms of the Jacobi polynomials, one can show that, for large μ,

$$\Phi_\mu(\theta) = \frac{1}{\sqrt{2\pi}}\exp\{[i(2\mu+a+b)\theta-i\kappa\pi]/2\}, \tag{6.17}$$

where $\kappa=-a,a$ for $-\pi<\theta<0$ and $0<\theta<\pi$, respectively. For large N, we obtain again (6.14) with $\gamma(\theta)=[(a+b)\theta+a\pi]/2$, $[(a+b)\theta-a\pi]/2$ for $-\pi<\theta<0$ and $0<\theta<\pi$. Thus $R_1(\theta)$ is given by

$$\bar{\rho}(\theta) = \frac{1}{2\pi}+\frac{a+b}{\beta\pi N}-\frac{2}{\beta N}\sum_{n=-\infty}^{\infty}\{a\delta(\theta-2n\pi) + b\delta[\theta-(2n+1)\pi]\}, \tag{6.18}$$

where we set $\beta=2$. Since $R_1(\theta)$ cannot be negative, we replace δ by δ_N in (6.18) to obtain (4.20). Again, unfolding as above, we obtain $S_N(r)$ and $\mathbf{R}_n$.

For more general weight functions satisfying (4.15), it is shown in [21] that the polynomials ϕ_μ and the normalized functions Φ_μ are given, respectively, by

$$\phi_\mu(e^{i\theta}) = e^{i\mu\theta}\{[D(e^{i\theta})]^*\}^{-1}, \tag{6.19}$$

$$\Phi_\mu(\theta) = \exp[i\mu\theta + i\gamma(\theta)], \tag{6.20}$$

for large μ. These are generalizations of the above special examples with the general result for $\gamma(\theta)$ given by (4.14). When $V=O(1)$ [as opposed to $V=O(N)$], we can use (6.20) in (6.10) to obtain (6.14). Thus, the level density $R_1(\theta)$ is given by (4.13) and the unfolded correlation functions by (3.3), (3.4), and (3.10). Note that in all of these cases the potential V is weak and the density to the leading order is a constant; the correction terms needed for the universality of long-range correlations, may however contain δ-function-like terms at the zeros of $w(\theta)$ as in (4.20).

For strong potentials, $V=O(N)$, the leading term in the density will not be a constant and may also display banded behavior [15]. In these cases (6.19) may not be applicable for $\mu \approx N$ and then the above arguments fail. In such cases short-range fluctuations are expected to be universal but the universality breaks down for long-range fluctuations; see (5.5). To show this by the polynomial methods, the following integral representation of the polynomials, similar to those for the real line [14] may be useful:

$$\phi_\mu(z) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left(\prod_{j=1}^{\mu} (z - e^{i\theta_j}) \right) \times \mathcal{P}_{\mu,2}(\theta_1, \ldots, \theta_\mu) d\theta_1 \cdots d\theta_\mu = \overline{\det(z-U)}, \tag{6.21}$$

where we consider monic polynomials (i.e., coefficient of the leading term is 1) and in the last form the average is over the corresponding ensemble of μ-dimensional unitary matrices U. Proof of (6.21) is outlined in Appendix E. We do not pursue here (6.21) for further investigation.

VII. POLYNOMIAL METHOD FOR THE ORTHOGONAL ENSEMBLES (EVEN DIMENSION)

For Dyson ensemble with $\beta=1$, the correlation functions were expressed in terms of $e^{ip\theta}$, where $p=-(N-1)/2, -(N-3)/2, \ldots, (N-1)/2$ [1]. Thus half-integral powers of $e^{i\theta}$ were used for even N while integer powers were needed for odd N. In this section, we consider the even case. For weight function $w(\theta)$, we define the skew-orthogonal functions $\phi_p(e^{i\theta})$ as

$$\int\int \phi_p(e^{i\theta_1})\phi_q(e^{i\theta_2})w(\theta_1)w(\theta_2)\epsilon(\theta_1-\theta_2)d\theta_1 d\theta_2 = g_p \delta_{p+q,0}, \tag{7.1}$$

where the ϵ function is defined in (3.9), g_p is the normalization constant with $g_{-p}=-g_p=g_p^*$ (pure imaginary), and $p,q = \pm 1/2, \pm 3/2, \ldots$. Here $\phi_p(e^{i\theta})$ involves the functions $e^{ik\theta}$ where $k=-p+1, -p+2, \ldots, p$. Thus, for $p>0$, $e^{i(p-1)\theta}\phi_p(e^{i\theta})$ is a polynomial in $z=e^{i\theta}$ of order $2p-1$ and $\phi_{-p}(e^{i\theta}) = [\phi_p(e^{i\theta})]^*$. We will refer to $\phi_p(e^{i\theta})$ as skew-orthogonal polynomial. For $w(\theta)=1$, $\phi_p(e^{i\theta})=e^{ip\theta}$, and $g_p=2\pi i/p$. It can be shown (by a Gram-Schmidt construction) that the $\phi_p(e^{i\theta})$ are uniquely defined if the coefficient of $e^{ip\theta}$ is fixed.

We define skew-orthonormal functions $\Phi_p(\theta)$ and their integrals $\Psi_p(\theta)$,

$$\Phi_p(\theta) = (g_{|p|})^{-1/2} w(\theta) \phi_p(e^{i\theta}), \tag{7.2}$$

$$\Psi_p(\theta) = \int \epsilon(\theta - \xi)\Phi_p(\xi)d\xi, \tag{7.3}$$

respectively. Then the corresponding skew-orthonormality condition will be

$$\int \Phi_p(\theta)\Psi_q(\theta)d\theta = \text{sign}(p)\delta_{p+q,0}. \tag{7.4}$$

We define the kernels,

$$S_N(\theta_1,\theta_2) = \sum_{p=1/2}^{(N-1)/2} [\Phi_p(\theta_1)\Psi_{-p}(\theta_2) - \Phi_{-p}(\theta_1)\Psi_p(\theta_2)], \tag{7.5}$$

$$D_N(\theta_1,\theta_2) = -\sum_{p=1/2}^{(N-1)/2} [\Phi_p(\theta_1)\Phi_{-p}(\theta_2) - \Phi_{-p}(\theta_1)\Phi_p(\theta_2)] = -\frac{\partial S_N(\theta_1,\theta_2)}{\partial \theta_2}, \tag{7.6}$$

$$I_N(\theta_1,\theta_2) = \sum_{p=1/2}^{(N-1)/2} [\Psi_p(\theta_1)\Psi_{-p}(\theta_2) - \Psi_{-p}(\theta_1)\Psi_p(\theta_2)] = \int \epsilon(\theta_1 - \xi) S_N(\xi,\theta_2)d\xi, \tag{7.7}$$

and

$$S_N^\dagger(\theta_1,\theta_2) = S_N(\theta_2,\theta_1). \tag{7.8}$$

We also define,

$$J_N(\theta_1,\theta_2) = -\sum_{p=(N+1)/2}^{\infty} [\Psi_p(\theta_1)\Psi_{-p}(\theta_2) - \Psi_{-p}(\theta_1)\Psi_p(\theta_2)] = I_N(\theta_1,\theta_2) - \epsilon(\theta_1-\theta_2). \tag{7.9}$$

In terms of quaternion kernel,

$$\sigma_{N,1}(\theta_1,\theta_2) = \begin{pmatrix} S_N(\theta_1,\theta_2) & D_N(\theta_1,\theta_2) \\ J_N(\theta_1,\theta_2) & S_N^\dagger(\theta_1,\theta_2) \end{pmatrix}, \tag{7.10}$$

the jpd (2.1) can be written as a quaternion determinant,

$$\mathcal{P}_{N,1}(\theta_1,\ldots,\theta_N) = \frac{1}{N!}\text{Qdet}[\sigma_{N,1}(\theta_j,\theta_k)]_{j,k=1,\ldots,N}. \tag{7.11}$$

The quaternion kernel satisfies the Dyson's conditions

$$\int \sigma_{N,1}(\theta,\theta)d\theta = N \quad (7.12)$$

and

$$\int \sigma_{N,1}(\theta_1,\theta_2)\sigma_{N,1}(\theta_2,\theta_3)d\theta_2 = \sigma_{N,1}(\theta_1,\theta_3) + \lambda\sigma_{N,1}(\theta_1,\theta_3) - \sigma_{N,1}(\theta_1,\theta_3)\lambda, \quad (7.13)$$

where

$$\lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (7.14)$$

Thus the n-level correlation function can be written as

$$R_{n,1}(\theta_1,\ldots,\theta_n) = \mathrm{Qdet}[\sigma_{N,1}(\theta_j,\theta_k)]_{j,k=1,\ldots,n}. \quad (7.15)$$

Proof of (7.11)–(7.13) is similar to the corresponding proof for Dyson ensembles [1,17] and is outlined in Appendix C. Note that, for $w(\theta)=1$, $\Psi_p(\theta)=\Phi_p(\theta)/ip$, giving thereby the earlier Dyson results.

We now consider asymptotic-N results for weak periodic potentials $[V(\theta)=O(1)]$. We propose the ansatz that for polynomials of asymptotic order ($|p|\gg 1$),

$$\Phi_p(\theta) = \sqrt{\frac{|p|}{2\pi i}}\exp\{i[p\theta + 2\gamma(\theta)]\}, \quad (7.16)$$

and then

$$\Psi_p(\theta) = \frac{1}{p}\sqrt{\frac{i|p|}{2\pi}}\exp\{i[p\theta + 2\gamma(\theta)]\}. \quad (7.17)$$

From (7.9), we have

$$S_N(\theta_1,\theta_2) = \delta(\theta_1-\theta_2) - \lim_{L\to\infty}\sum_{p=(N+1)/2}^{L}[\Phi_p(\theta_1)\Psi_{-p}(\theta_2) - \Phi_{-p}(\theta_1)\Psi_p(\theta_2)], \quad (7.18)$$

and therefore, for large N, using (7.16) and (7.17) in (7.18) and summing the geometrical series, we obtain

$$S_N(\theta_1,\theta_2) = \frac{\sin\left(\dfrac{N(\theta_1-\theta_2)}{2} + 2[\gamma(\theta_1)-\gamma(\theta_2)]\right)}{2\pi\sin[(\theta_1-\theta_2)/2]}. \quad (7.19)$$

Note that the δ function in (7.18) is cancelled by a similar term in the second term. The corresponding large-N results for D_N and I_N are differentials and integrals of S_N as in (7.6) and (7.7). Thus, the level density for weak potentials is given by

$$R_1(\theta) = S_N(\theta,\theta) = \frac{N}{2\pi} + \frac{2}{\pi}\frac{\partial\gamma}{\partial\theta}, \quad (7.20)$$

consistent with (4.13). With r_1, r_2 defined by (2.4) and (7.20) and $r=r_1-r_2$, we obtain the results (3.7)–(3.10),

$$S_N(r) = \left(\frac{2\pi}{N}\right)S_N(\theta_1,\theta_2), \quad (7.21)$$

$$D_N(r) = \left(\frac{2\pi}{N}\right)^2 D_N(\theta_1,\theta_2), \quad (7.22)$$

$$I_N(r) = I_N(\theta_1,\theta_2) \quad (7.23)$$

$$\epsilon(r) = \epsilon(\theta_1-\theta_2), \quad (7.24)$$

confirming thereby (3.3) and (3.5) for $\beta=1$.

Our ansatz is consistent with the result $[\phi_p=e^{ip\theta}, \gamma(\theta)=0]$ for $V(\theta)=0$. For a nontrivial example, we consider the weight function $w(\theta)=|e^{i\theta}+c|^{-2}$ with c a constant. In this case, we find that for $p\geq 3/2$, the (monic) polynomial ϕ_p, the corresponding integrated function Ψ_p, and the normalization constant g_p are given, respectively, by

$$\phi_p(e^{i\theta}) = \frac{\exp[i(p-1)\theta]}{c^*}|\exp(i\theta)+c|^2 - \frac{c}{c^*}A_p\exp[i(p-1)\theta]\times[\exp(-i\theta)+c^*], \quad (7.25)$$

$$\sqrt{g_{|p|}}\Psi_p(\theta) = \frac{\exp[i(p-1)\theta]}{i(p-1)c^*} - \frac{A_p}{c^*}\sum_{k=0}^{\infty}\left(\frac{-1}{c}\right)^k\frac{\exp[i(p+k-1)\theta]}{i(k+p-1)}, \quad (7.26)$$

$$g_p = \frac{2\pi i}{(p-1)|c|^2}(1-A_p). \quad (7.27)$$

Here

$$A_p^{-1} = \sum_{k=0}^{\infty}\frac{p-1}{(k+p-1)|c|^{2k}}. \quad (7.28)$$

We also have for $p=1/2$,

$$\phi_{1/2}(e^{i\theta}) = e^{i\theta/2}, \quad (7.29)$$

$$g_{1/2} = \frac{2\pi i}{|c|(|c|^2-1)}\ln\frac{|c|+1}{|c|-1}. \quad (7.30)$$

For $p\gg 1$, we obtain $A_p=1-|c|^{-2}$, $g_p=2\pi i/p|c|^4$ and thereby a confirmation of (7.16) where $e^{2i\gamma(\theta)}=(c^*+e^{-i\theta})/(c+e^{i\theta})$.

As in (6.21), we have again an integral representation [14] of the skew-orthogonal polynomials,

$$\phi_p(e^{i\theta}) = e^{-i(p-1)\theta}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left(\prod_{j=1}^{\mu}(e^{i\theta}-e^{i\theta_j})\right)\times\mathcal{P}_{\mu,1}(\theta_1,\ldots,\theta_\mu)d\theta_1\cdots d\theta_\mu = e^{-i(p-1)\theta}\overline{\det(e^{i\theta}-U)}, \quad (7.31)$$

where $\mu=2p-1$ and we have used monic polynomials. In the last form, the average is over the corresponding ensemble of μ-dimensional symmetric unitary matrices U. Proof of (7.31) is outlined in Appendix E.

VIII. POLYNOMIAL METHOD FOR THE ORTHOGONAL ENSEMBLES (ODD DIMENSION)

For odd N, we again consider $\phi_p(e^{i\theta})$ where $e^{i(p-1)\theta}\phi_p(e^{i\theta})$ is polynomial of order $2p-1$ for $p>0$ and $\phi_{-p}(e^{i\theta}) = [\phi_p(e^{i\theta})]^*$. Here p is integer and $\phi_p(e^{i\theta})$ consists of $e^{ik\theta}$ with $k=-p+1,-p+2,\ldots,p$. The skew orthogonality is defined by (7.1) for all $p,q(=0,\pm1,\pm2,\ldots)$ except $p=q=0$. Note that the polynomial ϕ_0 remains unpaired and its normalization g_0 is not determined by (7.1). We introduce an extra orthogonality condition [14,16],

$$\int \phi_0(e^{i\theta})\phi_p(e^{i\theta})w(\theta)d\theta = g_0\delta_{0,p}, \tag{8.1}$$

to fix the normalization g_0 of ϕ_0. Note that g_0 is real but the g_p for $p\neq 0$ are imaginary. We define $\Phi_p(\theta) = (g_{|p|})^{-1/2}w(\theta)\phi_p(e^{i\theta})$ as in (7.2) and

$$\Psi_p(\theta) = \int \epsilon(\theta-\xi)\Phi_p(\xi)d\xi + c_p, \tag{8.2}$$

where the constant c_p is given by

$$c_p = -\frac{\int\int \epsilon(\theta-\xi)\phi_p(e^{i\xi})w(\xi)w(\theta)d\xi d\theta}{\sqrt{g_{|p|}}\int w(\theta)d\theta}, \tag{8.3}$$

so that

$$\int \Phi_0(\theta)\Psi_p(\theta)d\theta = 0. \tag{8.4}$$

In general $c_p\neq 0$, but c_0 is necessarily zero, and moreover (7.4) remains valid for all p,q except $p=q=0$.

We introduce additional kernels [14,16]

$$M(\theta_1,\theta_2) = \Phi_0(\theta_1), \quad M^\dagger(\theta_1,\theta_2) = \Phi_0(\theta_2), \tag{8.5}$$

$$\mu(\theta_1,\theta_2) = \Psi_0(\theta_1), \quad \mu^\dagger(\theta_1,\theta_2) = \Psi_0(\theta_2). \tag{8.6}$$

Instead of (7.10), we use the quaternion kernel,

$$\sigma_{N,1}(\theta_1,\theta_2) = \begin{pmatrix} S_N(\theta_1,\theta_2) + M(\theta_1,\theta_2) & D_N(\theta_1,\theta_2) \\ J_N(\theta_1,\theta_2) + \mu(\theta_1,\theta_2) - \mu^\dagger(\theta_1,\theta_2) & S_N^\dagger(\theta_1,\theta_2) + M^\dagger(\theta_1,\theta_2) \end{pmatrix}, \tag{8.7}$$

where $S_N, D_N, J_N, S_N^\dagger$ are given in (7.5)–(7.9) with summation over $p=1,\ldots,(N-1)/2$. Then results (7.11)–(7.13) are valid here also (see Appendix C) and therefore the correlation function is given by (7.15). For $w(\theta)=1$, skew-orthogonal functions and their normalizations for $p\neq 0$ have the same form as in the preceding section, and, with $p=0$, we have $\Phi_0(\theta)=1/\sqrt{2\pi}, \Psi_0(\theta)=\theta/\sqrt{2\pi}, g_0=2\pi$. Equation (8.7) is thus consistent with the corresponding Dyson results.

For the weight function $w(\theta)=|e^{i\theta}+c|^{-2}$, the polynomials and their normalizations g_p for $p\geq 2$ are given by (7.25)–(7.28). We also have $\phi_0(e^{i\theta})=1$, $\phi_1(e^{i\theta})=[\exp(i\theta)+1/c^*]$, and $g_0=2\pi/(|c|^2-1)$, $g_1=2\pi i|c|^{-2}\ln[|c|^2/(|c|^2-1)]$. For $p\gg 1$, we obtain $g_p=2\pi i/p|c|^4$, thereby confirming the ansatz (7.16) for the odd case. The matrix-integral representation (7.31) is valid also for odd N for all $p>0$.

IX. POLYNOMIAL METHOD FOR THE SYMPLECTIC ENSEMBLES

For $\beta=4$, we need $\phi_p(e^{i\theta})$ defined by the skew-orthogonality relation

$$\int [\phi_p'(e^{i\theta})\phi_q(e^{i\theta}) - \phi_p(e^{i\theta})\phi_q'(e^{i\theta})]w(\theta)d\theta = g_p\delta_{p+q,0}, \tag{9.1}$$

where $\phi_p'(e^{i\theta})=d\phi_p(e^{i\theta})/d\theta$, g_p is the normalization constant with $g_{-p}=-g_p=g_p^*$, and $\phi_p(e^{i\theta})$ is the complex conjugate of $\phi_p(e^{i\theta})$. As in Secs. VII and VIII, $e^{i(p-1)\theta}\phi_p(e^{i\theta})$ can be chosen to be polynomials of order $2p-1$ in $z=e^{i\theta}$ for $p>0$ and $\phi_{-p}(e^{i\theta})=[\phi_p(e^{i\theta})]^*$. For symplectic ensembles, we need to consider only the case of half-integral values of p. We can also write the skew-orthogonal condition as

$$\int [\Phi_p'(\theta)\Phi_q(\theta) - \Phi_p(\theta)\Phi_q'(\theta)]d\theta = \text{sign}(p)\,\delta_{p+q,0}, \tag{9.2}$$

where

$$\Phi_p(\theta) = \sqrt{\frac{w(\theta)}{g_{|p|}}}\phi_p(e^{i\theta}). \tag{9.3}$$

We define the kernels,

$$S_{2N}(\theta_1,\theta_2) = \sum_{p=1/2}^{N-1/2} [\Phi_p'(\theta_1)\Phi_{-p}(\theta_2) - \Phi_{-p}'(\theta_1)\Phi_p(\theta_2)], \tag{9.4}$$

$$D_{2N}(\theta_1,\theta_2) = -\sum_{p=1/2}^{N-1/2} [\Phi_p'(\theta_1)\Phi_{-p}'(\theta_2) - \Phi_{-p}'(\theta_1)\Phi_p'(\theta_2)] = -\frac{\partial S_{2N}(\theta_1,\theta_2)}{\partial\theta_2}, \tag{9.5}$$

$$I_{2N}(\theta_1,\theta_2) = \sum_{p=1/2}^{N-1/2} [\Phi_p(\theta_1)\Phi_{-p}(\theta_2) - \Phi_{-p}(\theta_1)\Phi_p(\theta_2)] = \int_{\theta_2}^{\theta_1} S_{2N}(\xi,\theta_2)d\xi, \quad (9.6)$$

$$S^{\dagger}_{2N}(\theta_1,\theta_2) = S_{2N}(\theta_2,\theta_1), \quad (9.7)$$

and

$$\sigma_{N,4}(\theta_1,\theta_2) = \begin{pmatrix} S_{2N}(\theta_1,\theta_2) & D_{2N}(\theta_1,\theta_2) \\ I_{2N}(\theta_1,\theta_2) & S^{\dagger}_{2N}(\theta_1,\theta_2) \end{pmatrix}. \quad (9.8)$$

Then jpd (2.1) can be written as

$$\mathcal{P}_{N,4}(\theta_1,\ldots,\theta_N) = \frac{1}{N!}\mathrm{Qdet}[\sigma_{N,4}(\theta_j,\theta_k)]_{j,k=1,\ldots,N}. \quad (9.9)$$

Also $\sigma_{N,4}$ satisfies the Dyson's conditions,

$$\int \sigma_{N,4}(\theta,\theta)d\theta = N, \quad (9.10)$$

and

$$\int \sigma_{N,4}(\theta_1,\theta_2)\sigma_{N,4}(\theta_2,\theta_3)d\theta_2 = \sigma_{N,4}(\theta_1,\theta_3). \quad (9.11)$$

Therefore we obtain the n-level correlation functions,

$$R_{n,4}(\theta_1,\ldots,\theta_n) = \mathrm{Qdet}[\sigma_{N,4}(\theta_j,\theta_k)]_{j,k=1,\ldots,n}. \quad (9.12)$$

Proof of (9.9)–(9.11) is similar to the corresponding proof for the potential free case [1,17] and is outlined in Appendix D. For $w(\theta)=1$, $\phi_p(e^{i\theta})=e^{ip\theta}$, $g_p=2\pi ip$, and then we recover the corresponding Dyson results [1,17].

Now, for weak potentials we propose the ansatz for the asymptotic polynomials $p \gg 1$ that

$$\Phi_p(\theta) = \frac{1}{\sqrt{4\pi i|p|}}\exp\{i[p\theta+\gamma(\theta)]\}, \quad (9.13)$$

and, then as in Sec. VII, we have for large N,

$$S_{2N}(\theta_1,\theta_2) = \frac{\sin\{N(\theta_1-\theta_2)+[\gamma(\theta_1)-\gamma(\theta_2)]\}}{4\pi\sin[(\theta_1-\theta_2)/2]}. \quad (9.14)$$

Thus the level density is

$$R_1(\theta) = S_{2N}(\theta,\theta) = \frac{N}{2\pi} + \frac{1}{2\pi}\frac{\partial\gamma}{\partial\theta}, \quad (9.15)$$

and $S_{2N}(2r)=(2\pi/N)S_{2N}(\theta_1,\theta_2)=\sin(2\pi r)/2N\sin(\pi r/N)$ confirming again the universality of (3.3) and (3.6). Here the unfolding function (5.6) is obtained from (2.4) and (9.15).

For a confirmation of our ansatz, we consider the weight function $w(\theta)=|e^{i\theta}+c|^{-2}$ with c a constant. In this case, we find that the polynomials and their normalizations g_p for $p \geq 3/2$ are given by

$$\phi_p(e^{i\theta}) = \left(\frac{\exp(-i\theta)+c^*}{c^*}\right)^2\exp(ip\theta) - \frac{2\pi i(2p-3)}{|c|^2c^*g_{p-1}}\phi_{p-1}(e^{i\theta}), \quad (9.16)$$

$$g_p = \frac{2\pi i}{|c|^4}\left(2p(1+|c|^2)-4-\frac{2\pi i(2p-3)^2}{|c|^2g_{p-1}}\right), \quad (9.17)$$

and $\phi_{1/2}(e^{i\theta})=e^{i\theta/2}$, $g_{1/2}=2\pi i/(|c|^2-1)$. For $p \gg 1$, we obtain $g_p=4\pi ip/|c|^2$ and therefore (9.13).

For the weight function $w(\theta)=1+\alpha\cos\theta$ also (where $-1<\alpha<1$) the skew-orthogonal polynomials can be worked out. We have, again for $p \geq 3/2$,

$$\phi_p(e^{i\theta}) = e^{ip\theta} - \alpha(p-1/2)\frac{g_{1/2}}{g_{p-1}}\phi_{p-1}(e^{i\theta}), \quad (9.18)$$

$$g_p = g_{1/2}\left(2p - \alpha^2(p-1/2)^2\frac{g_{1/2}}{g_{p-1}}\right), \quad (9.19)$$

and $\phi_{1/2}(e^{i\theta})=e^{i\theta/2}$, $g_{1/2}=2\pi i$. Note that for $p \gg 1$, we obtain $g_p=2\pi ip(1+\sqrt{1-\alpha^2})$ and therefore (9.13) where $e^{2i\gamma(\theta)} = [e^{i\theta}+\alpha^{-1}(1+\sqrt{1-\alpha^2})]/[e^{-i\theta}+\alpha^{-1}(1+\sqrt{1-\alpha^2})]$.

Finally, we have again an integral representation [14] of the skew-orthogonal (monic) polynomials,

$$\phi_p(e^{i\theta}) = e^{-i(p-1)\theta}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left(\prod_{j=1}^{\mu}(e^{i\theta}-e^{i\theta_j})^2\right) \times\mathcal{P}_{\mu,4}(\theta_1,\ldots,\theta_\mu)d\theta_1\cdots d\theta_\mu = e^{-i(p-1)\theta}\overline{\det(e^{i\theta}-U)^2}, \quad (9.20)$$

where $\mu=p-1/2$. In the last form, the average is for $\beta=4$ type ensemble. See Appendix E for proof of (9.20).

X. MONTE CARLO STUDY OF EIGENANGLE FLUCTUATIONS

In this section, we develop numerical techniques based on Monte Carlo (MC) and Langevin dynamics to simulate the circular ensembles. The methods are similar to those developed in [15] for non-Gaussian ensembles of Hermitian matrices and are based on Dyson's Brownian motion models of eigenvalues [1,13].

The jpd of eigenangles (2.1) can be written in the form

$$\mathcal{P}_{N,\beta}(\theta_1,\ldots,\theta_N) = ce^{-\beta W}, \quad (10.1)$$

where the potential W,

$$W(\theta_1,\ldots,\theta_N) = \sum_{l=1}^{N}V(\theta_l) - \sum_{j>k}\ln\left|\sin\frac{(\theta_j-\theta_k)}{2}\right|, \quad (10.2)$$

consists of a repulsive two-dimensional Coulomb (logarithmic) potential and a one-body periodic potential $V(\theta)$. Note that, for given weight function $w(\theta)$, the potential $V(\theta)$ is obtained from (2.2). Using a fictitious time τ, we can interpret the jpd (10.1) as the equilibrium ($\tau\to\infty$) density of the Langevin equations,

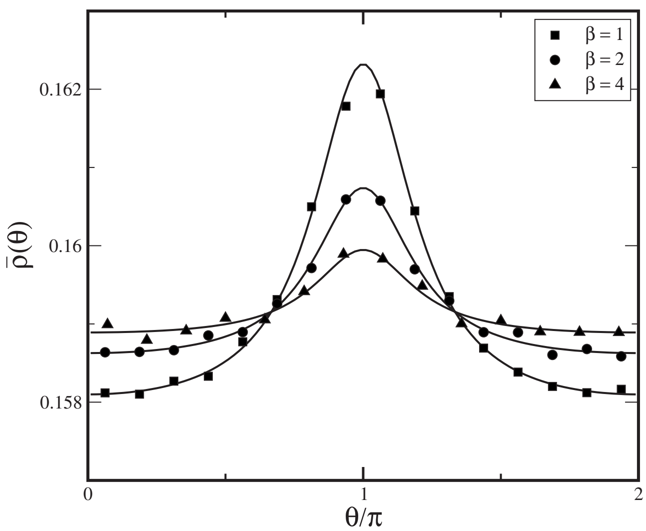

FIG. 1. $\bar{\rho}(\theta)$ vs θ for $w(\theta)=(5+4\cos\theta)^{-1}$. Solid lines are theoretical curves of (4.17) and the squares, circles and triangles denote the density from MC data for $\beta=1,2,4$, respectively.

$$\frac{d\theta_j}{d\tau} = -\frac{\partial W}{\partial \theta_j} + \xi_j(\tau). \quad (10.3)$$

Here the $\xi_j(\tau)$ are (real) uncorrelated Gaussian white noises with mean and covariance given, respectively, by

$$\bar{\xi}_j(\tau) = 0, \quad (10.4)$$

$$\overline{\xi_j(\tau_1)\xi_k(\tau_2)} = 2\beta^{-1}\delta(\tau_1-\tau_2)\delta_{jk}, \quad (10.5)$$

where the bar denotes ensemble averaging. Equation (10.3) can give an efficient method for generating eigenvalues in such ensembles, but one must carefully account for the unphysical level crossing induced by the discretization procedure. A faster method [15], which we follow here, relies on the stochastic Monte Carlo (MC) sampling of the eigenvalue space in the jpd (10.1).

For MC simulations of eigenangles we start with an initial set of θ_j variables $(j=1,\ldots,N)$ in the interval $[0,2\pi]$. A convenient choice for the initial spectra has constant spacing $2\pi/N$. A stochastic move assigns, to a randomly chosen θ_k, a new value θ_k' between 0 and 2π with a uniform probability. The move is accepted with probability $e^{-\beta\Delta W}$, where ΔW is the change in potential W resulting from the move. An MC step is defined as N moves, whether successful or not. Equilibrium is reached rapidly. In our calculations we find the equilibrium after 5000 MC steps for $N=201$ and then we choose spectrum after every 20 MC steps. We have analyzed ensembles of 10 000 spectra for the potentials discussed below.

As a check, we have done calculations for $V=0$ (viz., the Dyson ensembles) and found excellent agreement for $R_1(\theta)$ and $\Sigma^2(r)$ with the corresponding analytic results (3.1) and (3.18); the agreement is similar to the results for multiply kicked rotors in our earlier paper [20]. Here we show results for three examples of weak potentials: $w(\theta)=(5+4\cos\theta)^{-1}, |\sin\theta|$ and $|\sin 2\theta|$. The first of three weight functions corresponds to $c=2$ in (4.15) with $D(z)=(z+c)^{-1}$ and the density given by (4.17). The second is an example of the Jacobi weight function (4.18) with $a=b=1/2$ and the corresponding density is given by (4.20) with δ_N-like terms at the zeros of $w(\theta)$. The third weight function has four zeros and therefore four δ_N-like terms in the density. In this case, $D(z)=(1-z^4)^{1/2}/2$ and the constant term in the density is $(1+4/\beta N)/2\pi$. The MC results along with the theory are shown in Figs. 1, 2, and 3, respectively. To illustrate the effect of the δ_N terms, we have shown in Fig. 4 the cumulative density,

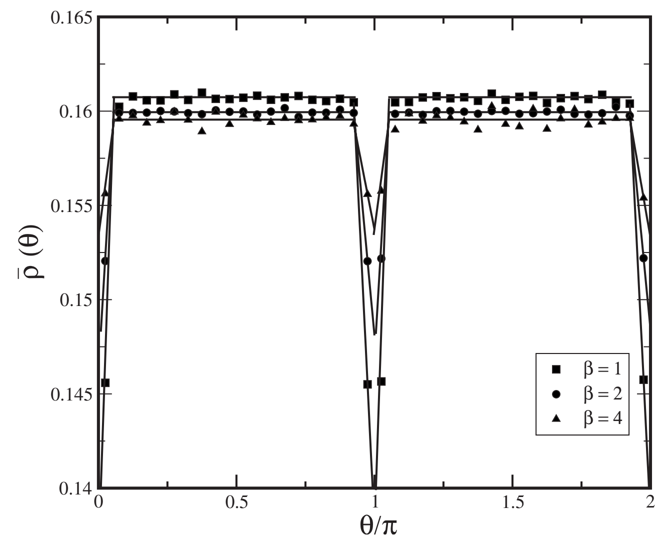

FIG. 2. $\bar{\rho}(\theta)$ vs θ for $w(\theta)=|\sin\theta|$. Note the dips in the density at $\theta=0,\pi,2\pi$ as predicted by (4.20).

$$\bar{F}(\theta) = \int_0^{\theta} \bar{\rho}(\theta')d\theta', \quad (10.6)$$

for $w(\theta)=|\sin\theta|$. Note that $\bar{F}(\theta)$ in this case is approximated well by two (closely spaced) parallel straight lines as in Fig. 4(b) instead of one straight line $[\bar{F}(\theta)=\theta/2\pi]$ as in Fig. 4(a). Indeed the theory as given in (4.20) predicts

$$\bar{F}(\theta)/N = \left(1+\frac{2(a+b)}{\beta N}\right)\frac{\theta}{2\pi} - \frac{a}{\beta N}, \quad 0<\theta<\pi,$$

$$= \left(1+\frac{2(a+b)}{\beta N}\right)\frac{\theta}{2\pi} - \frac{a+2b}{\beta N}, \quad \pi<\theta<2\pi, \quad (10.7)$$

and is in agreement with Fig. 4(b). For the third weight function $\bar{F}(\theta)$ will consist of four (closely spaced) parallel straight lines. For $\Sigma^2(r)$, we obtain departures from (3.18) for $r\approx N/2$ if we do unfolding by $r=N\theta/2\pi$, but we find excellent agreement with the theory if we use $r=N\bar{F}(\theta)$ to order one. We have shown the $\Sigma^2(r)$ results in Figs. 5 and 6 for the first two potentials.

In Figs. 7 and 8 we show densities for strong potentials $V(\theta)=\alpha N\cos\theta, \alpha N\ln(5+4\cos\theta)$, respectively. The theoretical prediction for the first potential is given by (4.28). For the second potential, the theoretical prediction is obtained by numerical integration of (4.21). In both cases, the agreement is excellent. Note that, for large α ($\alpha=5$ case in Figs. 7 and 8), the density becomes banded. We have checked the short-range fluctuations for $\Sigma^2(r)$ and found them to be in agreement with the corresponding result ($1 \lesssim r \ll N$) in (3.18). However, the long-range fluctuations show departures.

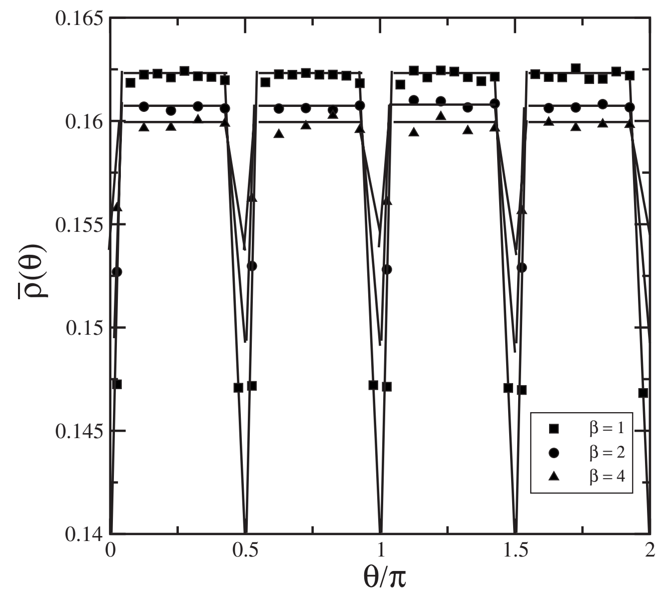

FIG. 3. $\bar{\rho}(\theta)$ vs θ for $w(\theta)=|\sin 2\theta|$. The dips in $\bar{\rho}(\theta)$ are at $\theta=0,\pi/2,\pi,3\pi/2,2\pi$ corresponding to the zeros of $w(\theta)$.

XI. UNIVERSAL CONDUCTANCE FLUCTUATIONS

In this section, we turn briefly to the conductance fluctuations in mesoscopic systems [5,7]. In particular we deal with the effect of non-uniformity of the ensemble on the universal conductance fluctuations (UCF) for quantum dots.

We consider the case of large number of incoming (N_1) and outgoing ($N_2=N-N_1$) channels; $N_1,N_2 \gg 1$. We take the above ensembles as ensembles of scattering matrices (U). In units of e^2/h the conductance can be written [7] as

$$g = \sum_{n=1}^{N_1} \sum_{m=N_1+1}^{N} |U_{nm}|^2. \tag{11.1}$$

We can also write it as

$$g = \mathrm{tr}(P_1 U P_2 U^\dagger) = N_1 - \mathrm{tr}(\Lambda A \Lambda^\dagger A) = \frac{N_1 N_2}{N} - \sum_{\mu,\nu} \exp[i(\theta_\mu - \theta_\nu)]|\delta A_{\mu\nu}|^2, \tag{11.2}$$

where P_1 and P_2 are, respectively, N_1- and N_2-dimensional projection matrices with matrix elements $P_1(j,k)=\delta_{jk}$ for $j \le N_1$ and zero otherwise and $P_2=I-P_1$. In the second step of (11.2) we have used the diagonal representation, $U=W\Lambda W^\dagger$, where Λ is the diagonalized form of U (i.e., $\Lambda_{\mu\nu}=e^{i\theta_\mu}\delta_{\mu\nu}$) and W is the unitary matrix that diagonalizes U (being orthogonal, unitary and symplectic for $\beta=1$, 2, and 4, respectively); we have also introduced $A=W^\dagger P_1 W$, which is P_1 matrix in the eigenbasis of U. In the last step, we have written the matrix A as $A=\langle A\rangle+\delta A$ where $\langle A\rangle=N^{-1}$ tr $A=N_1/N$. Note also that $\langle(\delta A)^2\rangle=N_1N_2/N^2$, as used below.

Since there is no correlation between eigenvalues and eigenvectors (as implied by the invariance of the ensembles), the ensemble averages over their distributions can be done separately. Thus the matrix elements $\delta A_{\mu\nu}$ are independent of the eigenangles. Moreover the jpd (2.1) is symmetric in all eigenangles, as also is the jpd of the matrix elements $\delta A_{\mu\nu}$; for example,

$$\langle(\delta A)^2\rangle = (N-1)\overline{|\delta A_{\mu\nu}|^2} + \overline{(\delta A_{\mu\mu})^2} \tag{11.3}$$

and

$$\overline{\left|\sum_\mu e^{i\theta_\mu}\right|^2} = N + N(N-1)\overline{e^{i(\theta_\mu-\theta_\nu)}}, \tag{11.4}$$

where $\mu \neq \nu$. Using the independence and symmetry of the jpd, we can write the ensemble average ($\bar{g}$) and variance [var(g)] of the conductance as

$$\bar{g} = N(1-\overline{e^{i(\theta_\mu-\theta_\nu)}})\left(\frac{N_1N_2}{N^2} - \overline{(\delta A_{\mu\mu})^2}\right), \tag{11.5}$$

and

$$\begin{aligned}
\mathrm{var}(g) &= (1-\overline{e^{i(\theta_\mu-\theta_\nu)}})^2\mathrm{var}\left(\sum_\mu (\delta A_{\mu\mu})^2\right) \\
&\quad + \sum_{\mu\neq\nu}\sum_{\mu'\neq\nu'}(\overline{e^{i(\theta_\mu-\theta_\nu)}e^{i(\theta_{\mu'}-\theta_{\nu'})}} - \overline{e^{i(\theta_\mu-\theta_\nu)}}\ \overline{e^{i(\theta_{\mu'}-\theta_{\nu'})}})\overline{|\delta A_{\mu\nu}|^2|\delta A_{\mu'\nu'}|^2} \\
&= N(1-\overline{e^{i(\theta_\mu-\theta_\nu)}})^2\{\mathrm{var}(\delta A_{\mu\mu})^2 + (N-1)\mathrm{cov}[(\delta A_{\mu\mu})^2,(\delta A_{\nu\nu})^2]\} + N^2(N-1)^2 \\
&\quad \times(\overline{e^{i(\theta_\mu-\theta_\nu)}e^{i(\theta_{\mu'}-\theta_{\nu'})}} - \overline{e^{i(\theta_\mu-\theta_\nu)}}\ \overline{e^{i(\theta_{\mu'}-\theta_{\nu'})}}) \\
&\quad \times\overline{|\delta A_{\mu\nu}|^2}\ \overline{|\delta A_{\mu'\nu'}|^2} \\
&\quad + N(N-1)(N-2)(N-3)(\overline{e^{i(\theta_\mu-\theta_\nu)}e^{i(\theta_{\mu'}-\theta_{\nu'})}} \\
&\quad - \overline{e^{i(\theta_\mu-\theta_\nu)}}\ \overline{e^{i(\theta_{\mu'}-\theta_{\nu'})}})\mathrm{cov}(|\delta A_{\mu\nu}|^2, \delta A_{\mu'\nu'}|^2) + N(N-1)(N-2)\mathrm{cov}(|\delta A_{\mu\nu}|^2,|\delta A_{\mu\nu'}|^2)(\overline{e^{i(2\theta_\mu-\theta_\nu-\theta_{\nu'})}} \\
&\quad + \overline{e^{-i(2\theta_\mu-\theta_\nu-\theta_{\nu'})}} + 2\overline{e^{i(\theta_\mu-\theta_\nu)}} - 4\overline{e^{i(\theta_\mu-\theta_\nu)}}\ \overline{e^{i(\theta_\mu-\theta_{\nu'})}}) \\
&\quad + N(N-1)[1+\overline{e^{2i(\theta_\mu-\theta_\nu)}} - 2(\overline{e^{i(\theta_\mu-\theta_\nu)}})^2]\mathrm{var}(|\delta A_{\mu\nu}|^2),
\end{aligned} \tag{11.6}$$

where in the last step (μ,ν) and (μ',ν') are distinct pairs of indices unlike the first step. Also, cov denotes covariance. Equations (11.5) and (11.6) are exact, valid for all N. Using the moments

$$\bar{m}_k = \int \exp(ik\theta)\bar{\rho}(\theta)d\theta, \tag{11.7}$$

and the averages [2]

$$\overline{\delta A_{\mu\mu}} = 0, \tag{11.8}$$

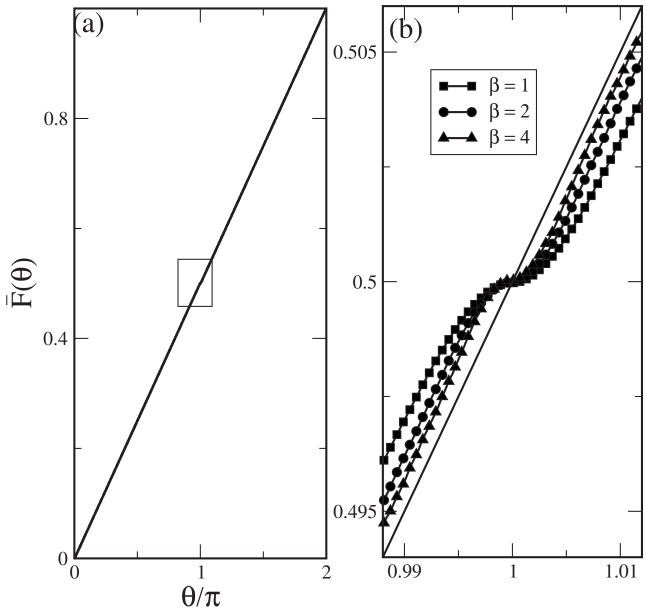

FIG. 4. $\bar{F}(\theta)$ vs θ for $w(\theta)=|\sin\,\theta|$. (a) The entire range of θ and (b) the inset box of (a).

$$\overline{(\delta A_{\mu\mu})^2} = \frac{2}{\beta N+2}\langle(\delta A)^2\rangle, \tag{11.9}$$

we have for large N,

$$\bar{g} = (1-|\bar{m}_1|^2)\frac{N_1 N_2}{N}. \tag{11.10}$$

Similarly [2], we have (again for large N),

$$\overline{|\delta A_{\mu\nu}|^2} = \frac{1}{N}\langle(\delta A)^2\rangle, \tag{11.11}$$

$$\overline{\delta A_{\mu\mu}\delta A_{\nu\nu}} = -\frac{2}{\beta N^2}\langle(\delta A)^2\rangle, \tag{11.12}$$

$$\mathrm{var}[(\delta A_{\mu\mu})^2] = 2\left(\frac{2}{\beta N}\langle(\delta A)^2\rangle\right)^2, \tag{11.13}$$

$$\frac{\mathrm{cov}[(\delta A_{\mu\mu})^2,(\delta A_{\nu\nu})^2]}{\mathrm{var}[(\delta A_{\mu\mu})^2]} = O\left(\frac{1}{N^2}\right), \tag{11.14}$$

$$\mathrm{var}(|\delta A_{\mu\nu}|^2) = \frac{2}{\beta N^2}\langle(\delta A)^2\rangle^2, \tag{11.15}$$

$$\frac{\mathrm{cov}(|\delta A_{\mu\nu}|^2,|\delta A_{\mu\nu'}|^2)}{\mathrm{var}(|\delta A_{\mu\nu}|^2)} = \frac{1}{N}\left(\frac{\langle(\delta A)^4\rangle}{\langle(\delta A)^2\rangle^2}-2\right) = \frac{1}{N}\left(\frac{N_1 N_2}{N^4}(N_1-N_2)^2-1\right), \tag{11.16}$$

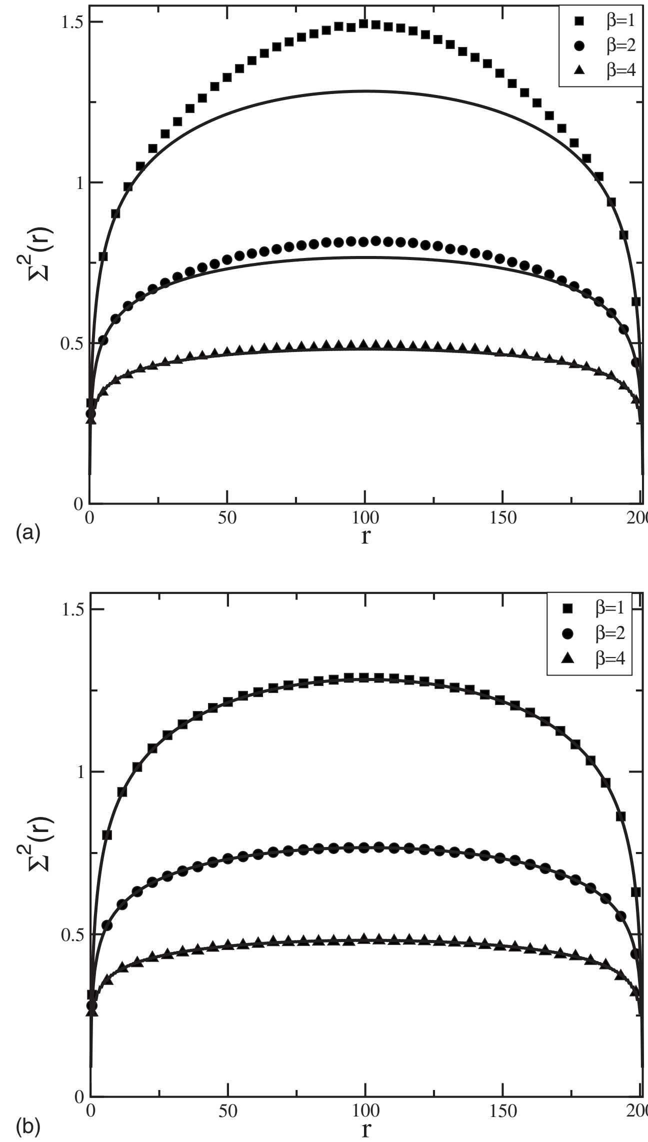

FIG. 5. $\Sigma^2(r)$ vs r for $w(\theta)=(5+4\cos\,\theta)^{-1}$ with unfolding (a) without the $O(N^{-1})$ term in $\bar{\rho}(\theta)$ and (b) with the $O(N^{-1})$ term. Agreement is found in (b) with theory (3.18) for all r.

$$\frac{\mathrm{cov}(|\delta A_{\mu\nu}|^2,|\delta A_{\mu'\nu'}|^2)}{\mathrm{var}(|\delta A_{\mu\nu}|^2)} = O\left(\frac{1}{N^2}\right), \tag{11.17}$$

so that var(g) is given by

$$\begin{aligned}\mathrm{var}(g) = {} & \left(\frac{2}{\beta}(1-2|\bar{m}_1|^2+2|\bar{m}_1|^4)+2|\bar{m}_1|^2 C_{1,-1}\right)\frac{N_1^2N_2^2}{N^4} \\ & -\frac{2}{\beta}[\bar{m}_2(\bar{m}_1^*)^2+\bar{m}_2^*(\bar{m}_1)^2-|\bar{m}_2|^2]\frac{N_1^2N_2^2}{N^4}+\frac{2}{\beta}[\bar{m}_2(\bar{m}_1^*)^2 \\ & +\bar{m}_2^*(\bar{m}_1)^2+2|\bar{m}_1|^2-4|\bar{m}_1|^4]\frac{N_1N_2}{N^4}(N_1-N_2)^2,\end{aligned} \tag{11.18}$$

where $C_{1,-1}$ is defined in (5.9).

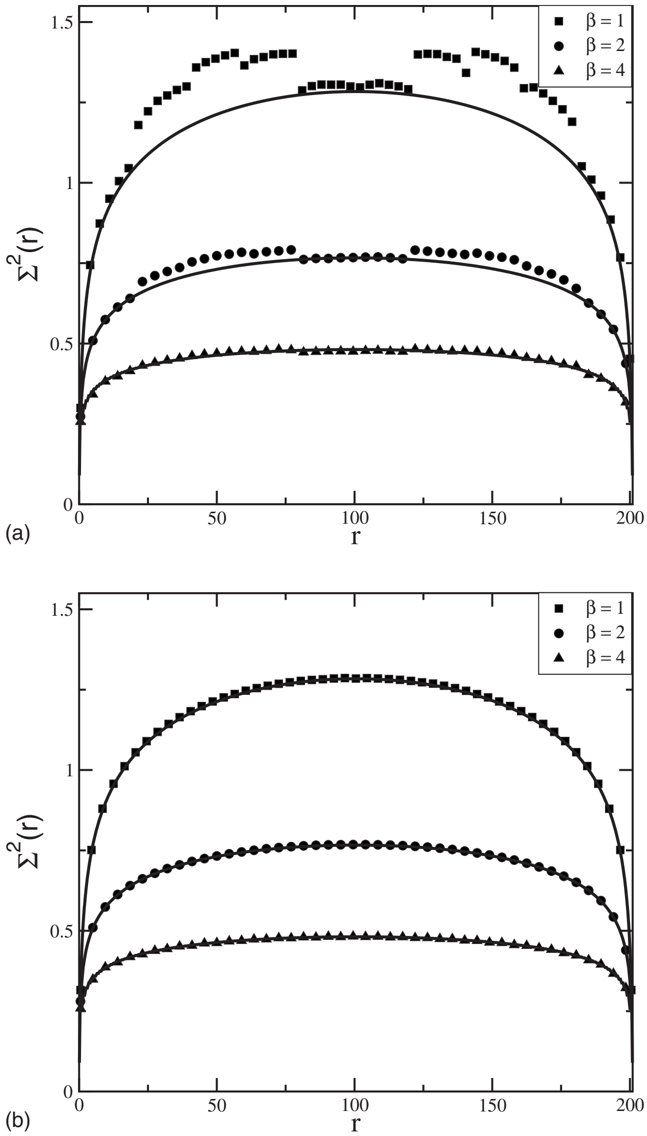

FIG. 6. $\Sigma^2(r)$ vs r for $w(\theta)=|\sin\,\theta|$ with unfolding in (a) and (b) as described in Fig. 5.

Note that for Dyson's circular ensemble the eigenangle density (3.1) is uniform and therefore $\bar{m}_p=\delta_{p0}$. Thus our results in (11.5) and (11.18) reproduce the average conductance $\bar{g}=N_1N_2/N$ and the universal (UCF) result $\mathrm{var}(g)=(2N_1^2N_2^2/\beta N^4)$, respectively [7]. Similarly for weak periodic potentials main part of the density continues to be uniform, giving thereby the same results to leading order in N. However for strong periodic potentials (banded as well as nonbanded cases) var(g) departs from the universality. Note that, from (5.13), $C_{1,-1}=2/\beta$ for weak potentials and also for the nonbanded case of strong potentials. But, whenever $\bar{m}_1\neq 0$, variance of g departs from the UCF. This breakdown of the universality requires separate investigation which we do not pursue here.

For Monte Carlo (MC) verification of the above results, we have generated ensemble of matrices U from the diagonal representation $U=W\Lambda W^{\dagger}$. Here we obtain Λ from the MC simulations of Sec. X, while W's are obtained as diagonalizing transformations of matrices belonging to the Gaussian ensembles. Thus, we can generate the scattering matrices corresponding to different weight functions. The right $N_1\times N_2$ upper block of the scattering matrix on multiplication with its Hermitian conjugate gives the transmission matrix. Let T_j be the transmission eigenvalues; then $g=\Sigma T_j$. We have studied the conductance for the potential $V(\theta)=\alpha N\cos\,\theta$ (as in Secs. IV and X) for several values of α. For this potential, $\bar{g}$ is given by (4.29) and (11.10). For var(g), we need $\bar{m}_2$ also,

$$\bar{m}_2=0,\quad 0\le\alpha\le 1/2,$$

$$=\left(1-\frac{1}{2\alpha}\right)^2,\quad 1/2\le\alpha. \tag{11.19}$$

Then for the nonbanded case ($0\le\alpha\le 1/2$) we have

$$\mathrm{var}(g)=\frac{2}{\beta}\left((1+2\alpha^4)\frac{N_1^2N_2^2}{N^4}+2\alpha^2(1-2\alpha^2)\frac{N_1N_2}{N^4}(N_1-N_2)^2\right), \tag{11.20}$$

while for the banded case ($\alpha\ge 1/2$) we obtain

$$\mathrm{var}(g)=\left[\frac{1}{\beta\alpha^2}\left(1-\frac{1}{2\alpha}+\frac{5}{64\alpha^2}\right)+2\left(\frac{1}{4\alpha}-1\right)^2C_{1,-1}\right]\frac{N_1^2N_2^2}{N^4}$$
$$+\frac{4}{\beta}\left\{\left(1-\frac{1}{2\alpha}\right)^2\left(1-\frac{1}{4\alpha}\right)^2+\left(1-\frac{1}{4\alpha}\right)^2\right.$$
$$\left.\times\left[1-2\left(\frac{1}{4\alpha}-1\right)^2\right]\right\}\frac{N_1N_2}{N^4}(N_1-N_2)^2. \tag{11.21}$$

We have done calculations with 50 000 scattering matrices of dimension $N=200$. The MC results for $\bar{g}$ and var(g) are shown in Figs. 9 and 10, respectively, and are consistent with our corresponding analytical results (11.10) and (4.29) and (11.20) and (11.21). Note that the first derivative of $\bar{g}$ is continuous at $\alpha=1/2$, showing thereby a smooth curve in Fig. 9. On the other hand, the first derivative of var(g) is discontinuous at $\alpha=1/2$ as seen in Fig. 10(a). Figure 11 gives the density of the transmission eigenvalues T for $\alpha=0$, 0.25, and 2.5 for $\beta=1$; almost same curves are obtained for $\beta=2$. The theory $\{\rho(T)=[\pi T(1-T)]^{-1/2}\}$ [7] is for $\alpha=0$. Departures from this are seen for $\alpha>0$ and in particular $\rho(T)=0$ for $T>T_c$ in the banded cases.

XII. CONCLUSION

Dyson introduced circular ensembles with uniform weight function as model for quantum chaotic systems. We have studied circular ensembles with nonuniform weight functions which are also relevant for quantum chaotic systems and are useful in other branches of physics.

We have shown that the level correlation functions can be written in terms of polynomials on the unit circle—orthogonal for $\beta=2$ and skew orthogonal for $\beta=1,4$. For $\beta=2$, we have used results from [21], whereas for $\beta=1,4$ we have worked out the skew-orthogonal polynomials for some weight functions. For weak periodic potentials, we study the asymptotic behavior of the polynomials and show that short-range and long-range fluctuations follow the universal result of uniform case. We elucidate the same result by the consideration of hierarchic relations among correlation functions and Monte Carlo simulations of the ensembles.

For strong potentials the density deviates from uniformity and in some cases develops banded behavior. We show this by analytic and Monte Carlo calculations. However, for these potentials, only the short-range fluctuations follow the universal behavior of the uniform case and not the long-range fluctuations.

Breakdown of universality for long-range fluctuations is also found in the study of conductance fluctuations. We have shown by analytic and numerical calculations that the circular ensembles give the UCF result for weak potentials but deviates from UCF for strong potentials.

ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF (3.18)

From (3.14) and (3.17) we note that

$$\begin{aligned}\Sigma^2(r) &= 2r\int_r^{N/2} Y_2(s)ds + 2\int_0^r sY_2(s)ds \\ &= \frac{2r}{\beta\pi N}\cot\left(\frac{\pi r}{N}\right) + 2\int_{r'}^{r} sY_2(s)ds + 2\int_0^{r'} sY_2(s)ds \\ &= \frac{2}{\beta\pi^2}\ln\left(\frac{\sin(\pi r/N)}{\sin(\pi r'/N)}\right) + \frac{2r'}{\beta\pi N}\cot\left(\frac{\pi r'}{N}\right) \\ &\quad + 2\int_0^{r'} sY_2(s)ds = \frac{2}{\beta\pi^2}\ln\tilde{r} - \frac{2}{\beta\pi^2}\ln(2\pi r') + \frac{2}{\beta\pi^2} \\ &\quad + 2\int_0^{r'} sY_2(s)ds = \frac{2}{\beta\pi^2}\ln\tilde{r} + \Sigma^2_{sr}(r') - \frac{2}{\beta\pi^2}\ln(2\pi r') \\ &= \frac{2}{\beta\pi^2}\ln\tilde{r} + C_\beta. \end{aligned} \tag{A1}$$

Here in the second step we have introduced $r'(\ll N)$ such that $1 \lesssim r' < r \leq N/2$, in the second and third steps we have used (3.13) for Y_2 in integrals with $s \geq r'$, in the fourth step we have replaced $\sin(\pi r'/N) \to \pi r'/N$, and in the fifth step we have used the short-range result [1,2]

$$\Sigma^2_{sr}(r') = \frac{2}{\beta\pi^2}\ln(2\pi r') + C_\beta. \tag{A2}$$

The constant C_β in (A1) and (A2) are $(\gamma+1)/\pi^2, 2(\gamma+1-\pi^2/8)/\pi^2$ and $(\ln 2+\gamma+1+\pi^2/8)/2\pi^2$, respectively, for $\beta=2,1,4$.

APPENDIX B: PROOF OF (6.3), (6.5), and (6.6)

For (6.3), we use the Vandermonde determinant [1]

$$\det[e^{i\mu\theta_\nu}]_{\substack{\mu=0,1,\ldots,N-1\\ \nu=1,2,\ldots,N}} = \prod_{j>k}(e^{i\theta_j} - e^{i\theta_k}). \tag{B1}$$

Since addition of a linear combination of rows does not affect the determinant, we can rewrite (B1) as

$$\det[\phi_\mu(e^{i\theta_\nu})]_{\substack{\mu=0,1,\ldots,N-1\\ \nu=1,2,\ldots,N}} \propto \prod_{j>k}(e^{i\theta_j} - e^{i\theta_k}). \tag{B2}$$

Multiplying the above equation by its complex conjugate and using Φ_μ of Sec. VI, we have

$$\begin{aligned}\prod_l w(\theta_l)\prod_{j>k}|e^{i\theta_j} - e^{i\theta_k}|^2 &= \det[\sqrt{w(\theta_\nu)}\,\phi_\mu(e^{i\theta_\nu})]^\dagger_{\substack{\mu=0,1,\ldots,N-1\\ \nu=1,2,\ldots,N}}\det[\sqrt{w(\theta_\nu)}\,\phi_\mu(e^{i\theta_\nu})]_{\substack{\mu=0,1,\ldots,N-1\\ \nu=1,2,\ldots,N}} \\ &= \det\left(\sum_{\mu=0}^{N-1}[\Phi_\mu(\theta_j)]^*\Phi_\mu(\theta_k)\right)_{j,k=1,2,\ldots,N} \propto \det[S_N(\theta_j,\theta_k)]_{j,k=1,2,\ldots,N}, \end{aligned} \tag{B3}$$

where † denotes the Hermitian conjugate. Equation (6.3) is thus obtained from (2.1) and (B3), the normalization constant being $(N!)^{-1}$. Equations (6.5) and (6.6) follow from the orthogonality condition (6.1).

APPENDIX C: PROOF OF (7.11)–(7.13) FOR EVEN AND ODD N

We first consider the even-N case of $\beta=1$. We start with proof of (7.11). For this we note [1] that for $\theta_1 \leq \theta_2 \leq \ldots. \leq \theta_N$,

$$\prod_{j>k}|\exp(i\theta_j) - \exp(i\theta_k)| = i^{-N(N-1)/2}\det[e^{ip\theta_j}] \propto \det[\phi_p(e^{i\theta_j})], \tag{C1}$$

where $p=-(N-1)/2, -(N-3)/2, \ldots, (N-1)/2$ and $j=1,2,\ldots,N$. Then we can write jpd (2.1) for $\beta=1$ as

$$\mathcal{P}_{N,\beta}(\theta_1,\ldots,\theta_N) \propto \det\begin{pmatrix}\Phi_{-p}(\theta_k)\\ \Phi_p(\theta_k)\end{pmatrix}, \tag{C2}$$

where row index $p=1/2, 3/2, \ldots, (N-1)/2$, column index $k=1,2,\ldots,N$, and the Φ_p are defined in (7.2).

Now, consider the $2N \times 2N$ matrix,

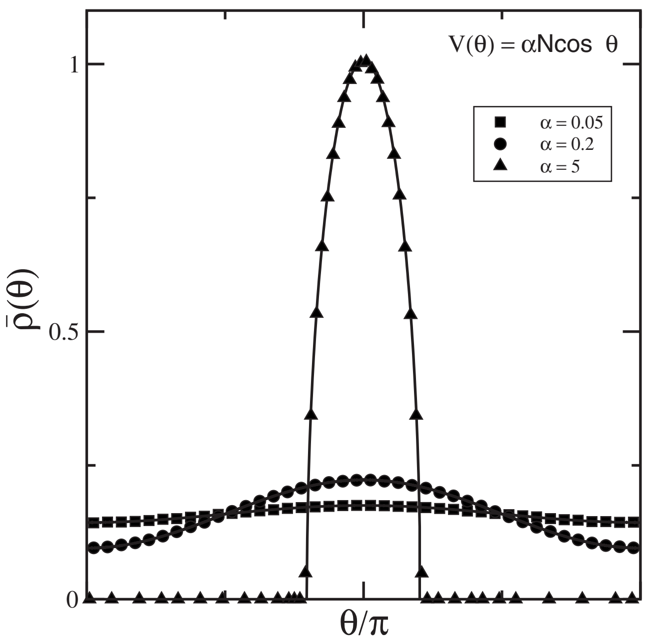

FIG. 7. $\bar{\rho}(\theta)$ vs θ for $V(\theta)=\alpha N\cos\theta$ with $\alpha=0.05, 0.2, 5$ for $\beta=1$. Solid lines show the analytic result (4.28).

$$G=\begin{bmatrix} S_N(\theta_j,\theta_k) & D_N(\theta_j,\theta_k)\\ I_N(\theta_j,\theta_k) & S_N^{\dagger}(\theta_j,\theta_k)\end{bmatrix}_{2N\times 2N} =\begin{pmatrix}\Phi_q(\theta_j) & \Phi_{-q}(\theta_j)\\ \Psi_q(\theta_j) & \Psi_{-q}(\theta_j)\end{pmatrix}_{2N\times N}\begin{pmatrix}\Psi_{-p}(\theta_k) & -\Phi_{-p}(\theta_k)\\ -\Psi_p(\theta_k) & \Phi_p(\theta_k)\end{pmatrix}_{N\times 2N}, \tag{C3}$$

where indices $p,q=1/2, 3/2, \ldots, (N-1)/2$ and j,k $=1, 2, \ldots, N$. From (C3), we see that the rank of the matrix G is N and hence the N rows $[I_N(\theta_j,\theta_k)\ \ S_N^{\dagger}(\theta_j,\theta_k)]$ of matrix G can be expressed as linear combination of the N rows $[S_N(\theta_j,\theta_k)\ \ D_N(\theta_j,\theta_k)]$. Thus for the quaternion matrix,

$$Q_{N1}=[\sigma_{N,1}(\theta_j,\theta_k)]_{j,k=1,2,\ldots,N}, \tag{C4}$$

the ordinary determinant of corresponding $2N\times 2N$ matrix is given by

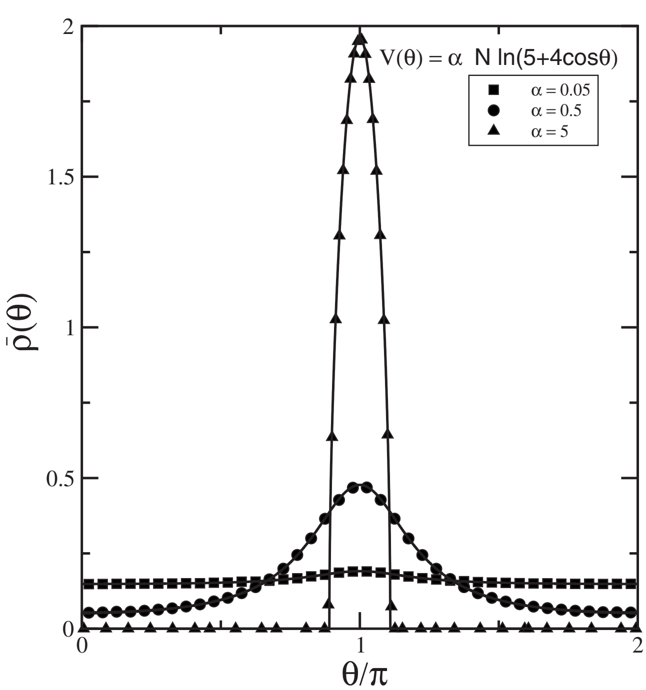

FIG. 8. $\bar{\rho}(\theta)$ vs θ for $V(\theta)=\alpha N\ln(5+4\cos\theta)$ with $\alpha=0.05, 0.5, 5$ for $\beta=1$. Solid lines are from numerical integration of (4.21).

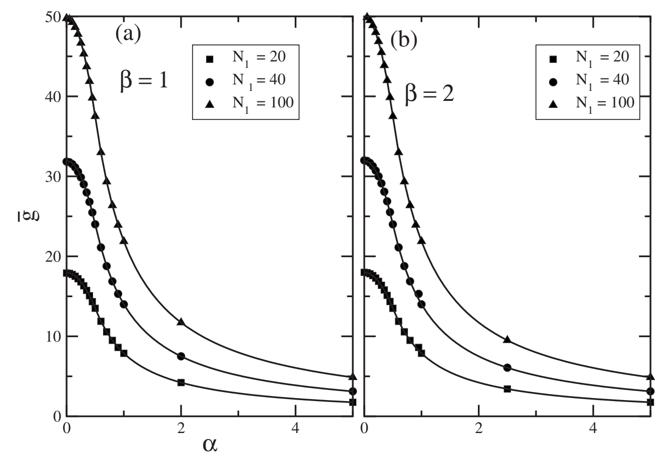

FIG. 9. $\bar{g}$ vs α for several values of N_1. (a) and (b) correspond, respectively, to $\beta=1, 2$.

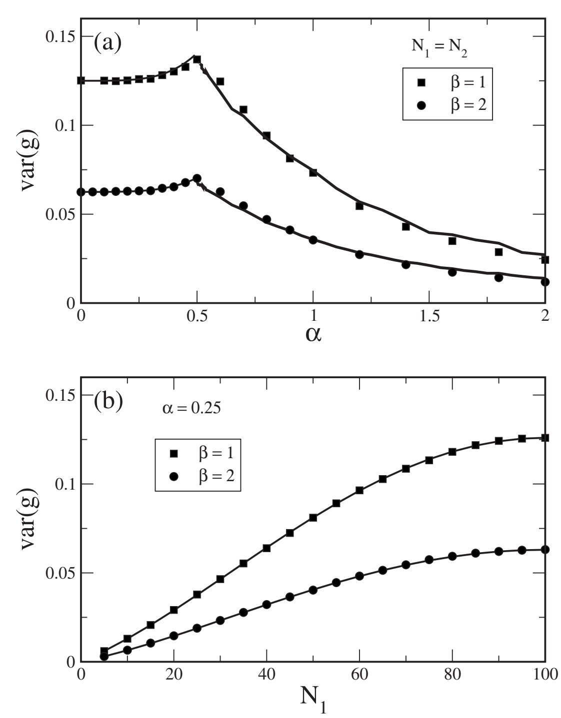

FIG. 10. (a) and (b) show var(g) vs α for $N_1=N_2$, and vs N_1 for $\alpha=0.25$, respectively. Solid squares and circles denote $\beta=1$ and 2, respectively. The solid lines correspond to the analytical results. In (a), we have used $C_{1,-1}=2/\beta$ for $0\le\alpha\le 1/2$, but for $\alpha>1/2$ we have estimated trace covariance $C_{1,-1}$ from the MC calculations. The agreement with (11.20) and (11.21) is good.

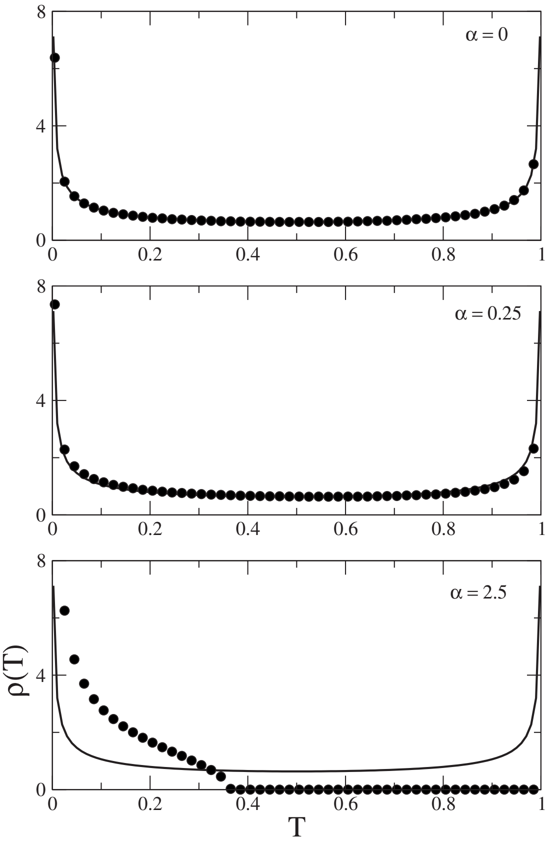

FIG. 11. Density of transmission eigenvalues vs T for $\beta=1, N_1=100$ with $V(\theta)=\alpha N\cos\theta$ and $\alpha=0, 0.25, 2.5$. Solid line corresponds to $1/\sqrt{\pi T(1-T)}$ [7].

$$\det[Q_{N1}] = \det\begin{bmatrix} S_N(\theta_j,\theta_k) & D_N(\theta_j,\theta_k) \\ J_N(\theta_j,\theta_k) & S_N^\dagger(\theta_j,\theta_k)\end{bmatrix} = \det\begin{bmatrix} S_N(\theta_j,\theta_k) & D_N(\theta_j,\theta_k) \\ -\epsilon(\theta_j-\theta_k) & 0\end{bmatrix} = \det[\epsilon(\theta_j - \theta_k)]\det[D_N(\theta_j-\theta_k)] \propto [\mathcal{P}_{N,1}(\theta_1,\theta_2,\ldots,\theta_N)]^2, \tag{C5}$$

as in the uniform case of COE [1]. We also note that the quaternion matrix, Q_{N1}, is self-dual. Thus square of the quaternion determinant (Qdet) is the corresponding ordinary determinant of (C5). This completes the proof of (7.11).

The proof of (7.12) and (7.13) follows from the skew-orthogonality condition (7.1). Eq. (7.12) can be verified directly, since $D_N(\theta,\theta)=0$ and $J_N(\theta,\theta)=0$. For (7.13), we note that

$$S_N * S_N = S_N,$$
$$D_N * S_N = S_N * D_N = D_N,$$
$$J_N * S_N = S_N * J_N = 0,$$
$$J_N * D_N = D_N * J_N = 0, \tag{C6}$$

where the $*$ operation is the convolution integral,

$$X * Y = \int X(\theta_1,\theta_2)Y(\theta_2,\theta_3)d\theta_2. \tag{C7}$$

Note here that the normalization in (7.11) is obtained from (7.12) and (7.13).

For odd N, the $2N\times 2N$ matrix G is defined as

$$G = \begin{bmatrix} S_N(\theta_j,\theta_k)+M(\theta_j,\theta_k) & D_N(\theta_j,\theta_k)+\delta M(\theta_j,\theta_k)M^\dagger(\theta_j,\theta_k) \\ I_N(\theta_j,\theta_k)+\delta^{-1} & S_N^\dagger(\theta_j,\theta_k)+M^\dagger(\theta_j,\theta_k)\end{bmatrix}_{2N\times 2N} = \begin{pmatrix} \sqrt{\delta}\Phi_0(\theta_j) & \Phi_q(\theta_j) & \Phi_{-q}(\theta_j) \\ \frac{1}{\sqrt{\delta}} & \Psi_q(\theta_j) & \Psi_{-q}(\theta_j)\end{pmatrix}_{2N\times N} \begin{pmatrix} \frac{1}{\sqrt{\delta}} & \sqrt{\delta}\Phi_0(\theta_k) \\ \Psi_{-p}(\theta_k) & -\Phi_{-p}(\theta_k) \\ -\Psi_p(\theta_k) & \Phi_p(\theta_k)\end{pmatrix}_{N\times 2N}. \tag{C8}$$

Here δ is arbitrary, $p,q=1,2,\ldots,(N-1)/2$ and $j,k=1,2,\ldots,N$. Thus the N rows $[I_N(\theta_j,\theta_k)+\delta^{-1}\;\; S_N^\dagger(\theta_j,\theta_k)+M^\dagger(\theta_j,\theta_k)]$ are linear combination of the N rows $[S_N(\theta_j,\theta_k)+M(\theta_j,\theta_k)\;\; D_N(\theta_j,\theta_k)+\delta M(\theta_j,\theta_k)M^\dagger(\theta_j,\theta_k)]$. Using this we have, for small δ,

$$\det\begin{bmatrix} S_N(\theta_j,\theta_k)+M(\theta_j,\theta_k) & D_N(\theta_j,\theta_k)+\delta\Phi_0(\theta_j)\Phi_0(\theta_k) \\ J_N(\theta_j,\theta_k)+\mu(\theta_j,\theta_k)-\mu^\dagger(\theta_j,\theta_k) & S_N^\dagger(\theta_j,\theta_k)+M^\dagger(\theta_j,\theta_k)\end{bmatrix} = \det\begin{bmatrix} S_N(\theta_j,\theta_k)+M(\theta_j,\theta_k) & D_N(\theta_j,\theta_k)+\delta\Phi_0(\theta_j)\Phi_0(\theta_k) \\ -\delta^{-1}-\epsilon(\theta_j-\theta_k)+\mu(\theta_j,\theta_k)-\mu^\dagger(\theta_j,\theta_k) & 0\end{bmatrix} = \det[D_N(\theta_j,\theta_k)+\delta\Phi_0(\theta_j)\Phi_0(\theta_k)]\det[\delta^{-1}+\epsilon(\theta_j-\theta_k)-\Psi_0(\theta_j)+\Psi_0(\theta_k)]. \tag{C9}$$

Here the first term is proportional to $\delta[\mathcal{P}_{N,1}(\theta_1,\ldots,\theta_N)]^2$ and the second term after subtracting the first column from each of the other columns is $\delta^{-1}\det[1+O(\delta),\epsilon(\theta_j-\theta_k)-\epsilon(\theta_j-\theta_1)-\Psi_0(\theta_j)+\Psi_0(\theta_k)]$, which is independent of the θ's in the leading order [1]. Thus, in the limit $\delta\to 0$, we obtain $\det[Q_{N1}]\propto(\mathcal{P}_{N,1})^2$, completing thereby the proof of (7.11) for odd N.

Proof of (7.12) follows from the skew-orthogonality condition (7.1) and the orthogonality condition (8.1). To verify (7.13), note that (C6) is valid for odd N also. The other integrals needed are

$$M*S_N=S_N*M=0,\quad \mu*S_N=S_N*\mu^\dagger=0,$$

$$M*D_N=D_N*M=0,\quad \mu*D_N=D_N*\mu^\dagger=0,$$

$$M*I_N=I_N*M=0,\quad M*M=M,$$

$$\epsilon*M=\mu*M=\mu,\quad \mu^\dagger*M=0, \tag{C10}$$

giving thereby (7.13) for odd N.

APPENDIX D: PROOF OF (9.9)–(9.11)

For $\beta=4$, we note that [1]

$$\prod_l w(\theta_l)\prod_{j>k}|\exp(i\theta_j)-\exp(i\theta_k)|^4 = \det\begin{pmatrix} e^{ip\theta_j} & pe^{ip\theta_j} \\ e^{-ip\theta_j} & -pe^{-ip\theta_j}\end{pmatrix}\prod_l w(\theta_l) \propto \det\begin{pmatrix}\Phi_p(\theta_j) & \Phi'_p(\theta_j)\\ \Phi_{-p}(\theta_j) & \Phi'_{-p}(\theta_j)\end{pmatrix}, \tag{D1}$$

where $p=1/2,3/2,\ldots,N-1/2$, $j=1,2,\ldots,N$, and the Φ_p are defined in Sec. IX. Then, for the $N\times N$ quaternion matrix

$$Q_{N4}=[\sigma_{N4}(\theta_j,\theta_k)]_{j,k=1,2,\ldots,N}, \tag{D2}$$

the ordinary $(2N\times 2N)$ determinant is

$$\det Q_{N4}=\det\left[\begin{pmatrix}\Phi'_p(\theta_j) & \Phi'_{-p}(\theta_j)\\ \Phi_p(\theta_j) & \Phi_{-p}(\theta_j)\end{pmatrix}\begin{pmatrix}\Phi_{-q}(\theta_k) & -\Phi'_{-q}(\theta_k)\\ -\Phi_q(\theta_k) & \Phi'_q(\theta_k)\end{pmatrix}\right] = \det\begin{pmatrix}\Phi'_p(\theta_j) & \Phi'_{-p}(\theta_j)\\ \Phi_p(\theta_j) & \Phi_{-p}(\theta_j)\end{pmatrix}^2 \propto [\mathcal{P}_{N,4}(\theta_1,\theta_2,\ldots,\theta_N)]^2. \tag{D3}$$

Here p and k are column indices while q and j are row indices. Using the fact that Q_{N4} is self-dual and working out the normalization constant we obtain (9.9).

Equation (9.10) follows directly from the skew-orthogonality condition (9.10), since $D_{2N}(\theta,\theta)=0$ and $I_{2N}(\theta,\theta)=0$. We also note that, with the definition (C7), we have

$$S_{2N}*S_{2N}=S_{2N},$$

$$D_{2N}*S_{2N}=S_{2N}*D_{2N}=D_{2N},$$

$$I_{2N}*S_N=S_{2N}*I_{2N}=0,$$

$$I_{2N}*D_{2N}=D_{2N}*I_{2N}=0, \tag{D4}$$

which give (9.11).

APPENDIX E: PROOF OF MATRIX-INTEGRAL REPRESENTATIONS

In this appendix we outline a proof of the matrix-integral representations (6.21), (7.31), and (9.20) of the polynomials. The proofs are similar to those for the polynomials on the real line; see Appendix B of [14].

We start with the $\beta=2$ case. To prove that the polynomials (6.21) are orthogonal, we have to show that, for $\nu=0,1,\ldots,\mu-1$,

$$\int e^{-i\nu\theta}\phi_\mu(e^{i\theta})w(\theta)d\theta=0. \tag{E1}$$

For notational convenience we write the Vandermonde determinant (B1) as

$$\Delta_N\equiv\Delta_N(e^{i\theta_1},e^{i\theta_2},\ldots,e^{i\theta_N})=\det[e^{i\mu\theta_\nu}]_{\substack{\mu=0,1,\ldots,N-1\\ \nu=1,2,\ldots,N}}. \tag{E2}$$

Then, the left-hand side (lhs) of (E1) is proportional to

$$\int d\theta_1\cdots d\theta_{\mu+1}e^{-i\nu\theta_{\mu+1}}\Delta_\mu(e^{-i\theta_1},\ldots,e^{-i\theta_\mu})\Delta_{\mu+1}(e^{i\theta_1},\ldots,e^{i\theta_{\mu+1}})\prod_{l=1}^{\mu+1}w(\theta_l) = \frac{1}{\mu+1}\int d\theta_1\cdots d\theta_{\mu+1}D^{(\nu)}_{\mu+1}(e^{-i\theta_1},\ldots,e^{-i\theta_{\mu+1}})\Delta_{\mu+1}(e^{i\theta_1},\ldots,e^{i\theta_{\mu+1}})\prod_{l=1}^{\mu+1}w(\theta_l), \tag{E3}$$

where $D^{(\nu)}_{\mu+1}$ is the determinant,

$$D^{(\nu)}_{\mu+1}(e^{i\theta_1},\ldots,e^{i\theta_{\mu+1}})=\det\begin{bmatrix}1 & 1 & \cdots & 1\\ e^{i\theta_1} & e^{i\theta_2} & \cdots & e^{i\theta_{\mu+1}}\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ e^{i(\mu-1)\theta_1} & e^{i(\mu-1)\theta_2} & \cdots & e^{i(\mu-1)\theta_{\mu+1}}\\ e^{i\nu\theta_1} & e^{i\nu\theta_2} & \cdots & e^{i\nu\theta_{\mu+1}}\end{bmatrix}_{(\mu+1)\times(\mu+1)}. \tag{E4}$$

The rhs of (E3) is obtained by symmetrization of the lhs. The determinant in (E4) is zero for $\nu=0,1,2,\ldots,\mu-1$ (since two rows of the matrix will be identical) but not for $\nu=\mu$, proving thereby (E1) and hence (6.21).

For $\beta=1$, we consider the even-N case; a similar proof will apply to the odd-N case for $p\neq 0$. The polynomials defined in (7.31) are skew orthogonal if

$$\int\int \epsilon(\theta-\xi)e^{-iq\xi}\phi_p(e^{i\theta})w(\theta)w(\xi)d\theta d\xi=0 \tag{E5}$$

for $q=p-1,p-2,\ldots,-p$ but not for $q=p$. The integral in (7.31) involve $|\Delta_\mu|$ and therefore Mehta's method of integration [1] over alternate variables can be used. The integral in (E5) is proportional to

$$\int d\theta_1\cdots d\theta_{\mu+2}\epsilon(\theta_{\mu+1}-\theta_{\mu+2})e^{-iq\theta_{\mu+2}}\prod_{\nu=1}^{\mu}(e^{i\theta_{\mu+1}}-e^{i\theta_\nu})|\Delta_\mu(e^{i\theta_1},\ldots,e^{i\theta_\mu})|\prod_{l=1}^{\mu+2}w(\theta_l)\propto\int_{\theta_1\le\cdots\le\theta_\mu\&\theta_{\mu+1}\le\theta_{\mu+2}}d\theta_1\ldots\ldots d\theta_{\mu+2}\epsilon(\theta_{\mu+1}$$
$$-\theta_{\mu+2})e^{-iq\theta_{\mu+2}-i(\mu-1)\Sigma_{\nu=1}^{\mu+1}\theta_\nu/2}\prod_{\nu=1}^{\mu}(e^{i\theta_{\mu+1}}-e^{i\theta_\nu})\Delta_\mu(e^{i\theta_1},\ldots,e^{i\theta_\mu})\prod_{l=1}^{\mu+2}w(\theta_l)$$
$$\propto\int d\theta_1 d\theta_3\cdots d\theta_{\mu+1}\prod_{l=0}^{\mu/2}w(\theta_{2l+1})\det\begin{bmatrix}e^{i(\mu+1)\theta_1/2} & F_{(\mu+1)/2}(\theta_1) & \cdots & e^{i(\mu+1)\theta_{\mu+1}/2} & F_{(\mu+1)/2}(\theta_{\mu+1})\\ e^{i(\mu-1)\theta_1/2} & F_{(\mu-1)/2}(\theta_1) & \cdots & e^{i(\mu-1)\theta_{\mu+1}/2} & F_{(\mu-1)/2}(\theta_{\mu+1})\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ e^{-i(\mu-1)\theta_1/2} & F_{-(\mu-1)/2}(\theta_1) & \cdots & e^{-i(\mu-1)\theta_{\mu+1}/2} & F_{-(\mu-1)/2}(\theta_{\mu+1})\\ e^{-iq\theta_1} & F_{-q}(\theta_1) & \cdots & e^{-iq\theta_{\mu+1}} & F_{-q}(\theta_{\mu+1})\end{bmatrix}, \tag{E6}$$

where $\mu=2p-1$ is even and

$$F_k(\theta)=\int_\theta^\pi e^{ik\xi}w(\xi)d\xi. \tag{E7}$$

In the last step of (E6) the integration over alternate variables is done and then the integrand symmetrized as in [14]. The determinant in (E6) vanishes for $q=p-1,p-2,\ldots,-p$, but not for $q=p$, proving thereby (E5) and hence (7.31).

For $\beta=4$, the polynomials defined in (9.20) are skew orthogonal if

$$\int[e^{-iq\theta}\phi_p'(e^{i\theta})+iqe^{-iq\theta}\phi_p(e^{i\theta})]w(\theta)d\theta=0, \tag{E8}$$

for $q=p-1,p-2,\ldots,-p$ but not for $q=p$. Using $\mu=p-1/2$, the lhs is proportional to

$$\int d\theta_1\cdots d\theta_{\mu+1}\left[\exp(-iq\theta_{\mu+1})\frac{\partial}{\partial\theta_{\mu+1}}\left(e^{-i(\mu-1/2)\theta_{\mu+1}}\prod_{j=1}^{\mu}(e^{i\theta_{\mu+1}}-e^{i\theta_j})^2\right)+iqe^{-iq\theta_{\mu+1}}e^{-i(\mu-1/2)\theta_{\mu+1}}\prod_{j=1}^{\mu}(e^{i\theta_{\mu+1}}-e^{i\theta_j})^2\right]$$
$$\times|\Delta_\mu(e^{i\theta_1},\ldots,e^{i\theta_\mu})|^4\left(\prod_{l=1}^{\mu+1}w(\theta_l)\right)$$

$$\propto \int d\theta_1 \cdots d\theta_{\mu+1} \left(\prod_{l=1}^{\mu+1} w(\theta_l) \right) \det \begin{bmatrix} e^{i(\mu+1/2)\theta_1} & i(\mu+1/2)e^{i(\mu+1/2)\theta_1} & \cdots & e^{i(\mu+1/2)\theta_{\mu+1}} & i(\mu+1/2)e^{i(\mu+1/2)\theta_{\mu+1}} \\ e^{i(\mu-1/2)\theta_1} & i(\mu-1/2)e^{i(\mu-1/2)\theta_1} & \cdots & e^{i(\mu-1/2)\theta_{\mu+1}} & i(\mu-1/2)e^{i(\mu-1/2)\theta_{\mu+1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ e^{-i(\mu-1/2)\theta_1} & -i(\mu-1/2)e^{-i(\mu-1/2)\theta_1} & \cdots & e^{-i(\mu-1/2)\theta_{\mu+1}} & -i(\mu-1/2)e^{-i(\mu-1/2)\theta_{\mu+1}} \\ e^{-iq\theta_1} & -iqe^{-iq\theta_1} & \cdots & e^{-iq\theta_{\mu+1}} & -iqe^{-iq\theta_{\mu+1}} \end{bmatrix}, \tag{E9}$$

where as in [14] we have used (D1) and then symmetrized the integrand. The skew-orthogonality condition (E8) follows from (E9), since the determinant vanishes for $q=p-1, p-2, \ldots, -p$.

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